INVERTED PENDULUM

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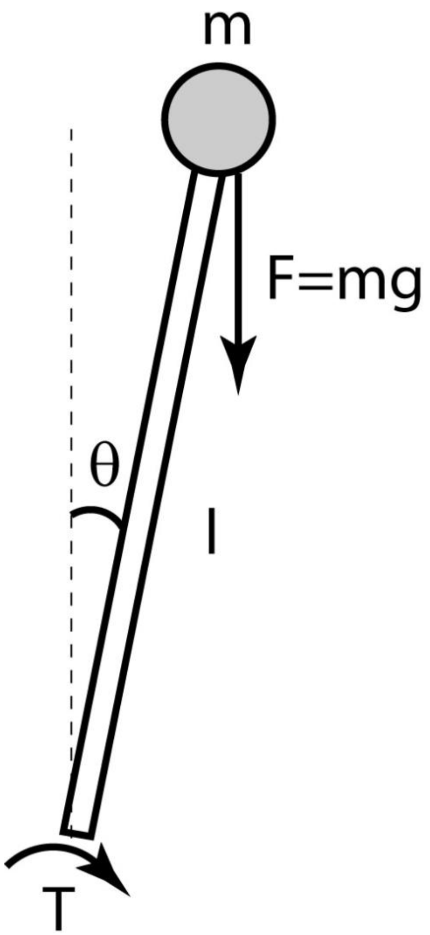
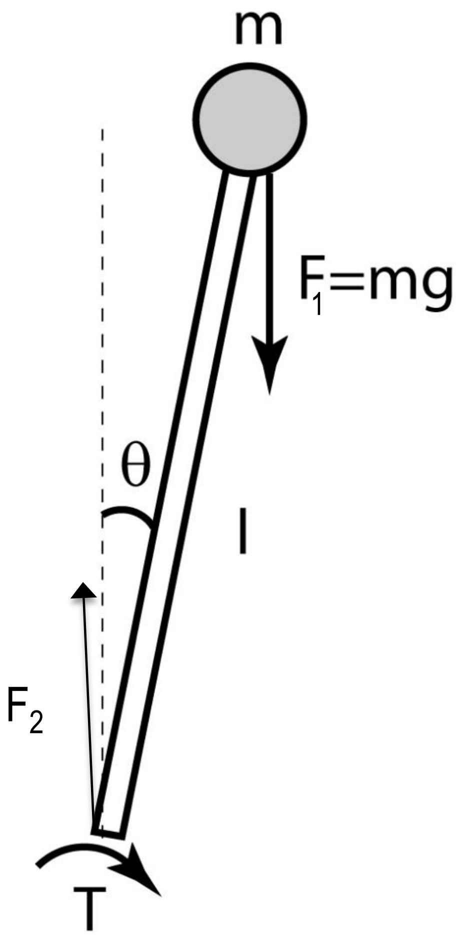
NatSciDemos on Youtube

WHAT'S HAPPENING?





nib Effect

Rotating Pulsar

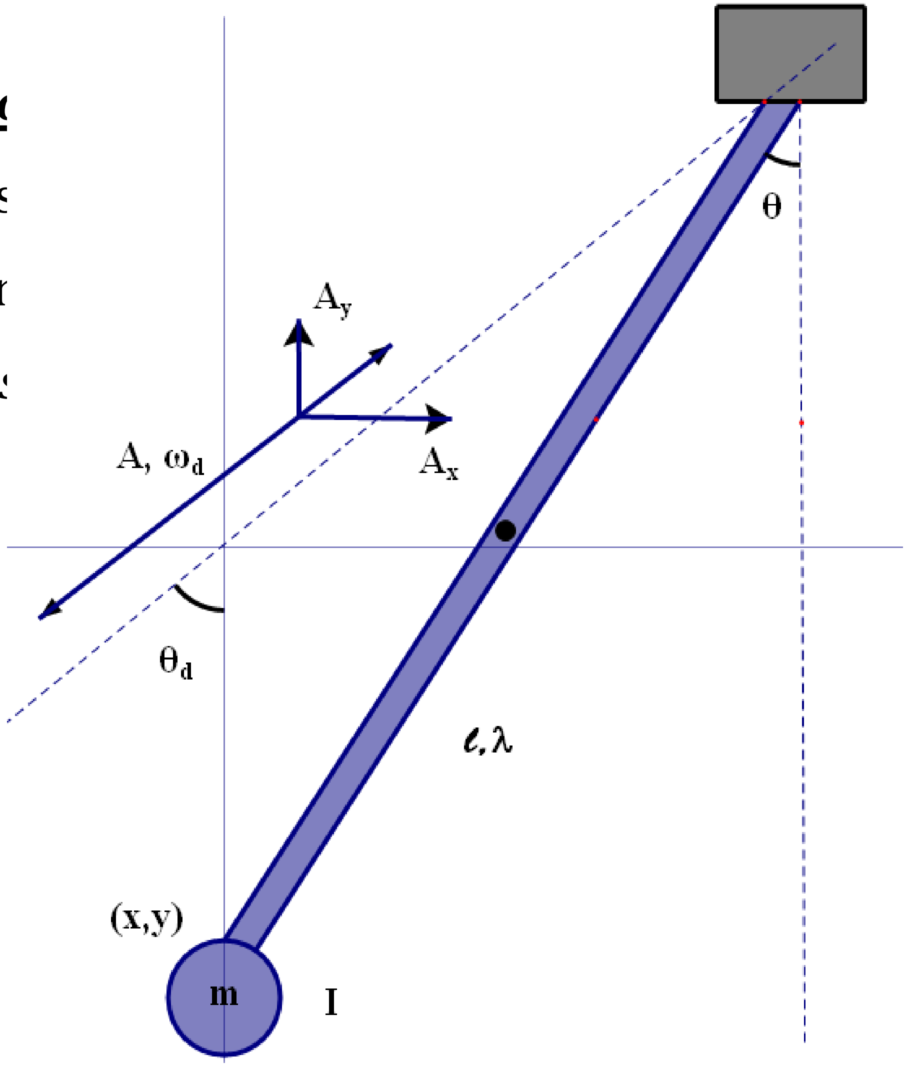
S

Applications

THE EQUATION OF MOTION

Position and Vel

- Center of mass
  - $x = -l\,\mathrm{si}$
  - $y = -lco$
- The velocity
  - $v_x = \dot{x} =$
  - $v_y = \dot{y} =$



THE EQUATION OF MOTION

Kinetic and Potential Energy

- Kinetic Energy

$$T = \underbrace{\frac{1}{2} I\dot{\theta}^2}_{\textbf{Rotational Kinetic Energy}} + \underbrace{\frac{1}{2} m\left(l^2\dot{\theta}^2 \cos^2\theta + A_x^2\omega_d^2 \cos^2(\omega_d t) - 2A_x\omega_d \cos(\omega_d t)\cos\theta\right)}_{\textbf{Translational Kinetic Energy for x}}$$

$$+ \underbrace{\frac{1}{2} m\left(l^2\dot{\theta}^2 \sin^2\theta + A_y^2\omega_d^2 \sin^2(\omega_d t) + 2A_y\omega_d \sin(\omega_d t)\sin\theta\right)}_{\textbf{Translational Kinetic Energy for y}}$$

- Potential Energy

$$V = mgy = mg(-l\cos\theta - A_y\cos(\omega_d t))$$

THE EQUATION OF MOTION

Euler-Lagrange Equations

- The Lagrangian is $\mathcal{L} = \mathrm{T} - \mathrm{V}$

- The equation of motion is obtained from the Euler-Lagrange Equation:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

$$\boldsymbol{\ddot{\theta} + \frac{g}{\lambda} sin\,\theta + \frac{A\omega_d^2}{\lambda} sin(\theta - \theta_d) cos(\omega_d t) = 0}$$

THE EQUATION OF MOTION

The Dimensionless Form

- Introduce non-dimensional time τ and dimensionless parameters γ and α where:

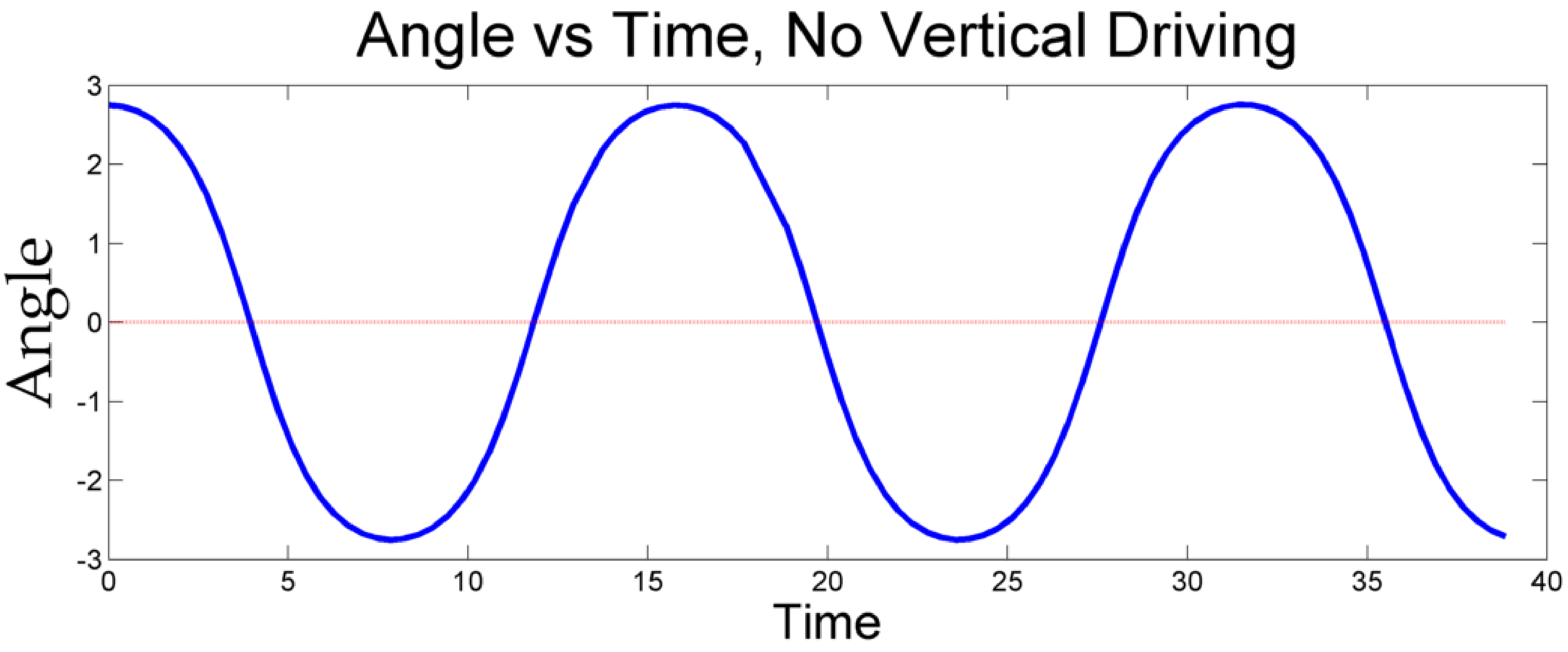
$$\tau = \omega_d t \qquad \gamma = \frac{\omega_o^2}{\omega_d^2} \qquad \alpha = \frac{A}{\lambda}$$

- The dimensionless equation of motion:

$$\frac{d^2\theta}{d\tau^2} + \gamma \sin\theta + \alpha\, sin(\theta - \theta_d) cos\, \tau = 0$$

NUMERICAL ANALYSIS

GENERIC PENDULUM

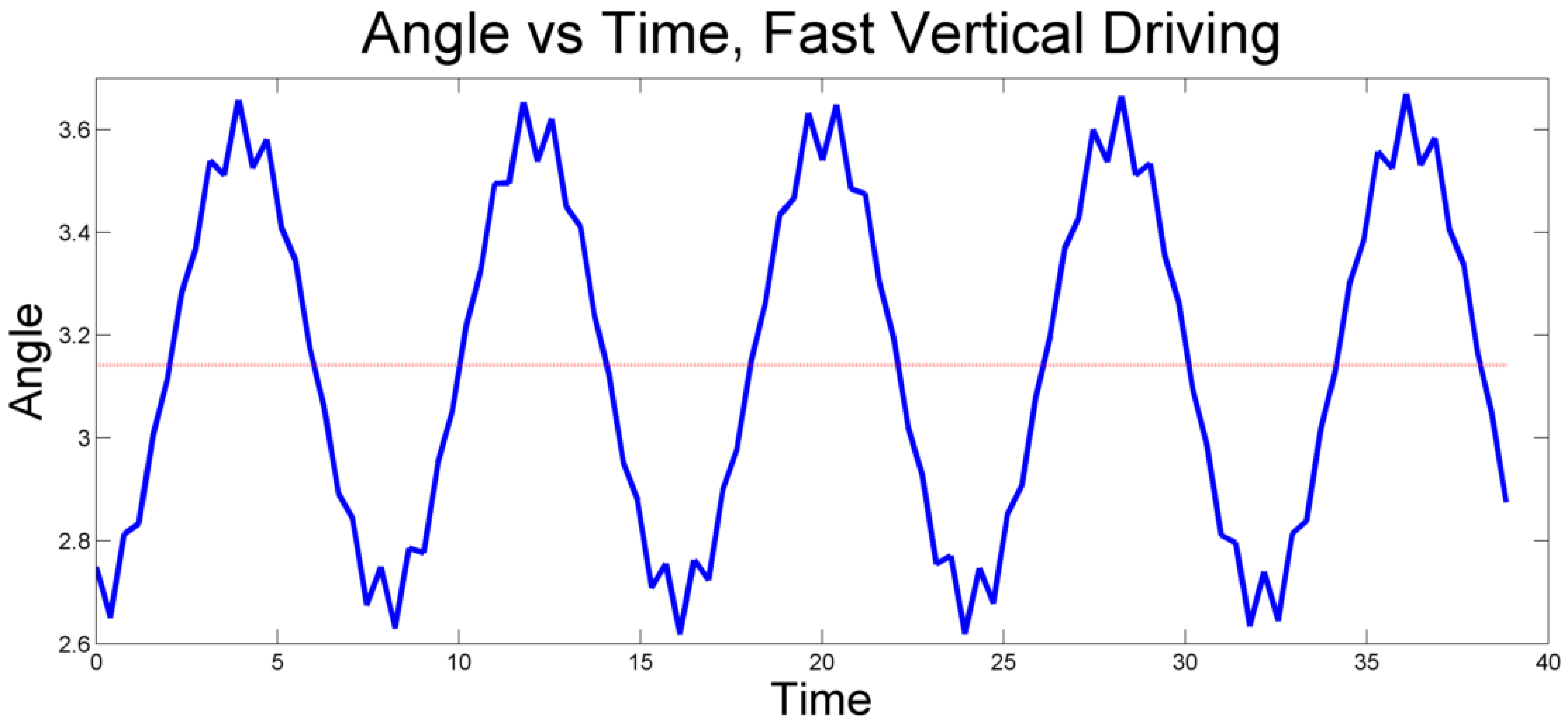


Initial angle: (7/8)π Radians

- Stability about: 0 Radians (Straight down)

NUMERICAL ANALYSIS

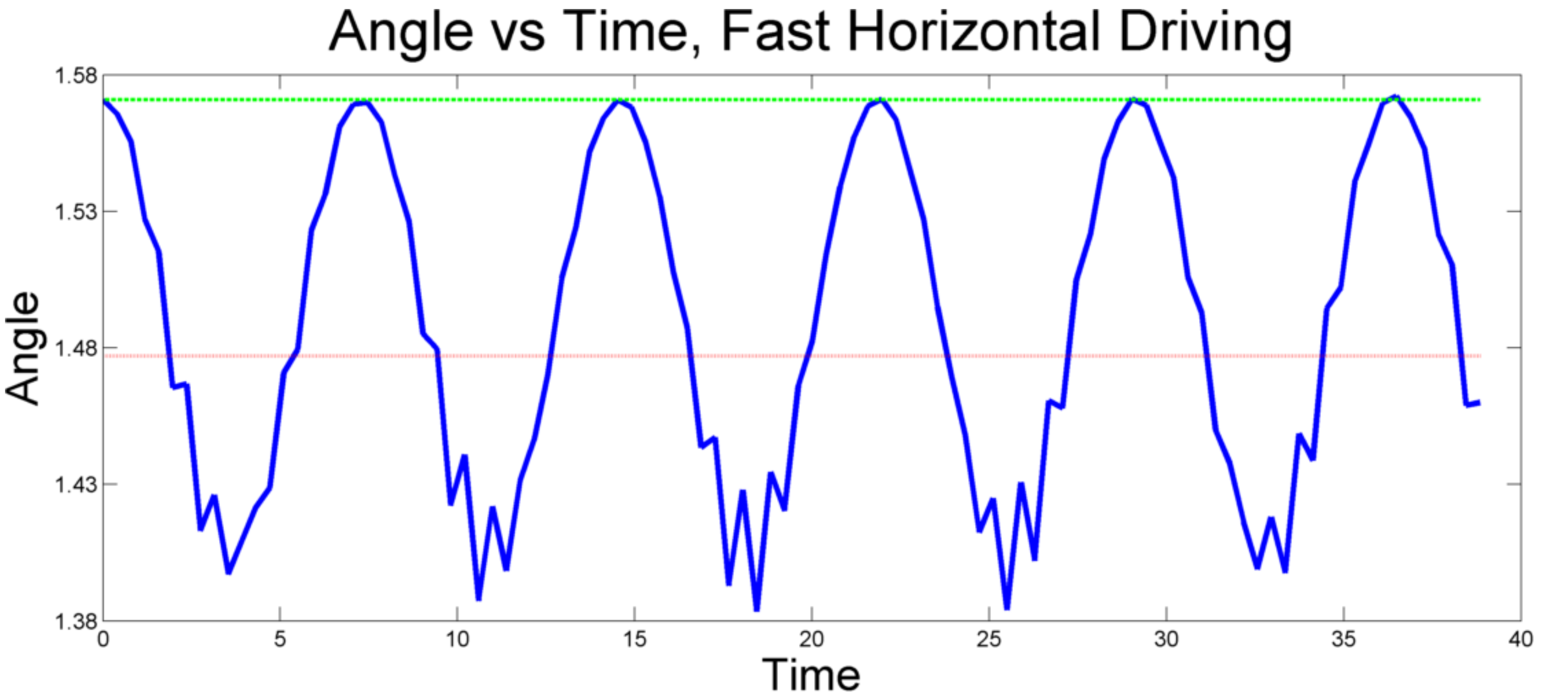
VERTICALLY DRIVEN JIGSAW



Initial angle: (7/8)π (Slightly left of vertical)
Drive angle: π (Vertical drive)
🟥 Stability about: π (Straight up)
Drive frequency: 314.16 drives per second (dps)

NUMERICAL ANALYSIS

HORIZONTALLY DRIVEN JIGSAW



Initial angle: (1/2)π (Horizontally left)

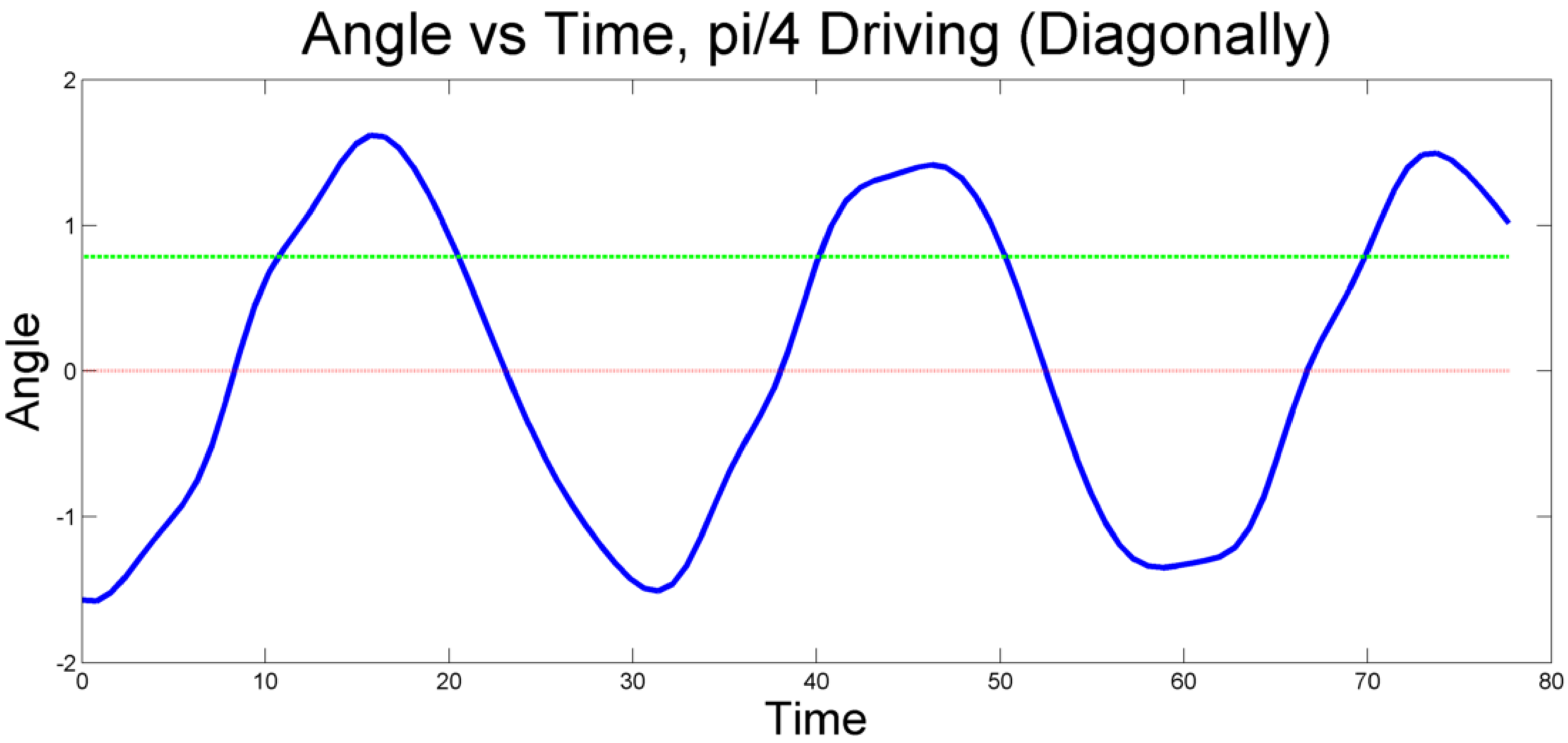
🟩 Drive angle: (1/2)π (Driven from the right)

🟥 Stability about: Approx. 1.477 Radians (Approximately 5.4 degrees below the horizontal)

Drive frequency: 314.16 drives per second

NUMERICAL ANALYSIS

DIAGONALLY DRIVEN JIGSAW



Initial angle: -(1/2)π (Horizontally right)

🟩 Drive angle: -(1/4)π (Driven from the bottom right)

🟥 Stability about: 2π (straight down, with odd behavior)

Drive frequency: 314.16 drives per second

AVERAGING & EFFECTIVE POTENTIAL U_{EFF}

CONCEPT

- Separate the *slow* and *fast* components

SLOW		*FAST*	
force of gravity	F (θ)	driving force	f (θ,t)
pendulum's rotation	Φ (t)	rapid driving oscillations	ξ (t)

- Assume the fast components have:
  - ***MUCH higher frequency*** *and*
  - ***relatively low amplitude***

- The *effective potential* U_{eff} is the average potential energy associated with Φ (t)

AVERAGING & EFFECTIVE POTENTIAL U_{EFF}

FINDING U_{EFF}

1. Relate the fast and slow components:

$$I_o\ddot{\theta} = I_o\left(\ddot{\phi} + \ddot{\xi}\right) = \left(F(\theta) + f(\theta, t)\right)$$

2. First-order approximation discarding insignificant terms:

$$I_o\left(\ddot{\phi} + \ddot{\xi}\right) \cong F(\phi) + \frac{dF}{d\theta}(\phi)\xi + f(\phi, t) + \frac{df}{d\theta}(\phi, t)\xi \longrightarrow I_o\ddot{\xi} \cong f(\phi, t)$$

3. Substitute back into and time-average the equation from 1.:

$$I_o\ddot{\phi} \cong F(\phi) + \langle\frac{df}{d\theta}(\phi, t)\xi\rangle$$

4. Use the definition of the effective potential:

$$\ddot{\phi} \cong -\frac{1}{I_o}\frac{dU_{eff}}{d\phi}$$

5. To solve for the effective potential:

$$U_{eff} = U_o - \int\left(\langle\frac{df}{d\theta}(\phi, t)\xi\rangle\right)d\phi$$

AVERAGING & EFFECTIVE POTENTIAL U_{EFF}

FINDING U_{EFF} (CONT.)

6. Write the effective potential in the terms from our governing equation:

$$F(\theta) = -I_o \gamma \sin\theta \qquad f(\theta,t) = -I_o \alpha\, D(\theta,t)$$

$$U_o = -\int F(\theta)d\theta = I_o \gamma \cos\theta$$

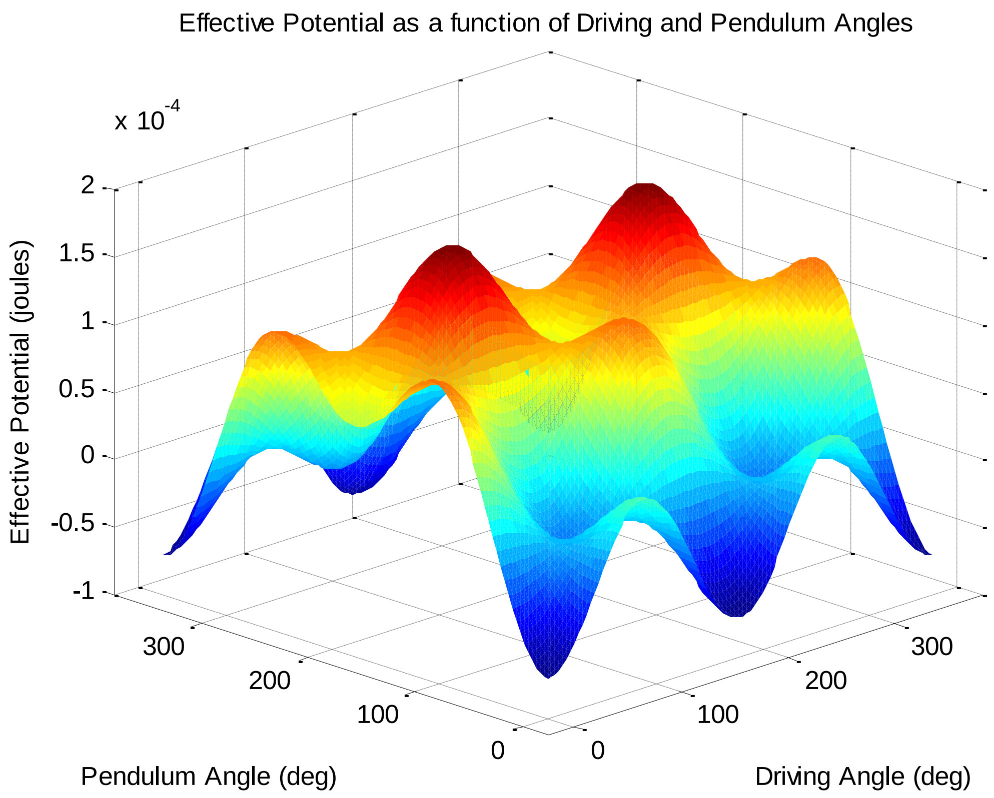
$$U_{eff} = U_o + \frac{\langle f^2 \rangle}{2I_o}$$

7. Finally, simplify to get:

$$\boldsymbol{U_{eff} = I_o\left(-\gamma\cos\theta + \frac{\alpha^2}{4}\left(\cos^2\theta_d \sin^2\theta + \sin^2\theta_d \cos^2\theta\right)\right)}$$

AVERAGING & EFFECTIVE POTENTIAL U_{EFF}

SURFACE PLOT



No Driving:

$I_o = 0.443$

$\gamma = 1$

$\alpha = 0$

With Driving:

$I_o = 0.443$

$\gamma = 1.6x10^{-5}$

$\alpha = 0.01$

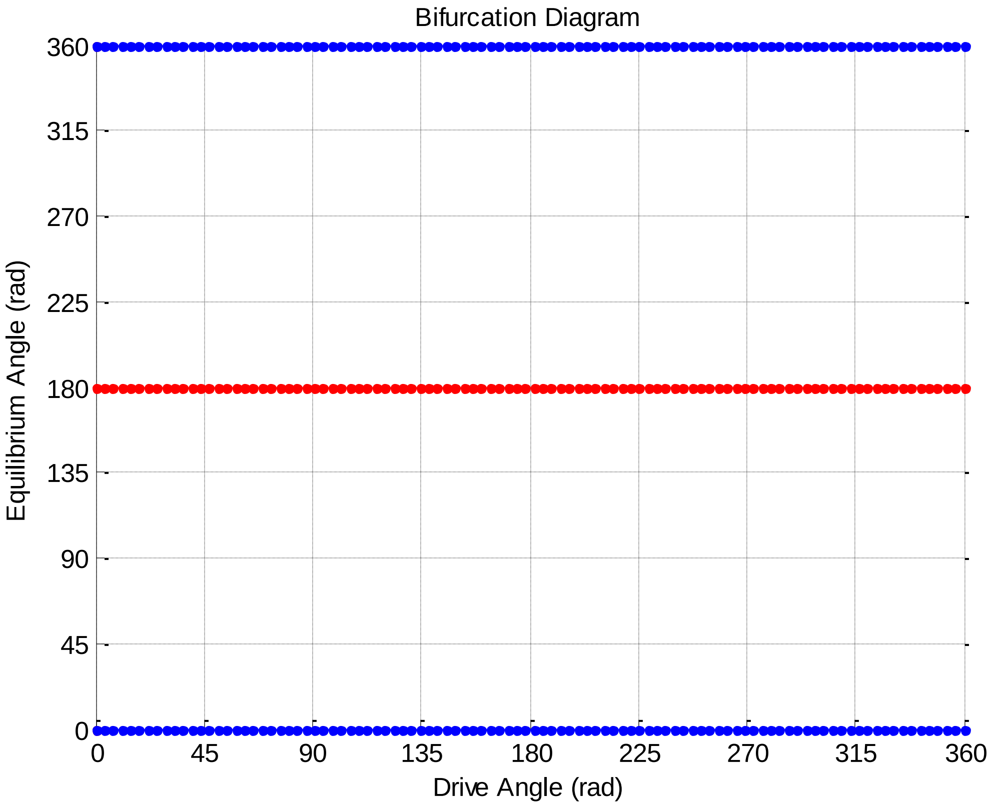
AVERAGING & EFFECTIVE POTENTIAL U_{EFF}

STABILITY ANALYSIS

Equilibrium (θ_{eq})		Stable	Unstable
$\theta_{eq} = 0$		$\gamma > \frac{-\alpha^2}{2} \cos 2\theta_d$	$\gamma < \frac{-\alpha^2}{2} \cos 2\theta_d$
$\theta_{eq} = \pi$		$\gamma < \frac{\alpha^2}{2} \cos 2\theta_d$	$\gamma > \frac{\alpha^2}{2} \cos 2\theta_d$
$\theta_{eq} = \pm\, arccos\left(\frac{-2\gamma}{\alpha^2\, cos\, 2\theta_d}\right)$	$\cos 2\theta_d > 0$	$\gamma^2 > \frac{\alpha^4 cos^2(2\theta_d)}{4}$	
	$\cos 2\theta_d = 0$	Always stable	
	$\cos 2\theta_d < 0$	$\gamma^2 < \frac{\alpha^4 cos^2(2\theta_d)}{4}$	

AVERAGING & EFFECTIVE POTENTIAL U_{EFF}

BIFURCATION DIAGRAM

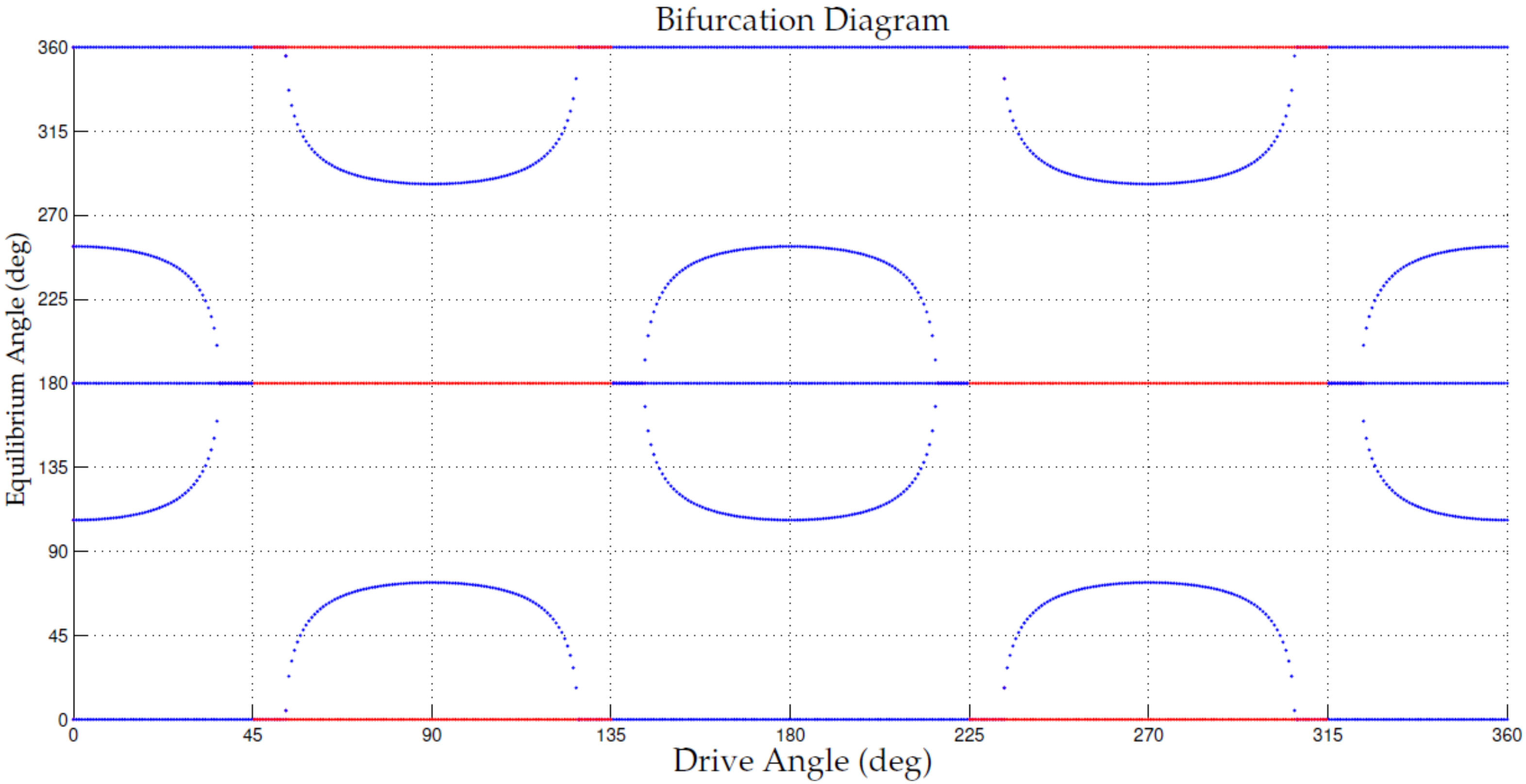


No Driving

- ● stable
- ● unstable

AVERAGING & EFFECTIVE POTENTIAL U_{EFF}

BIFURCATION DIAGRAM



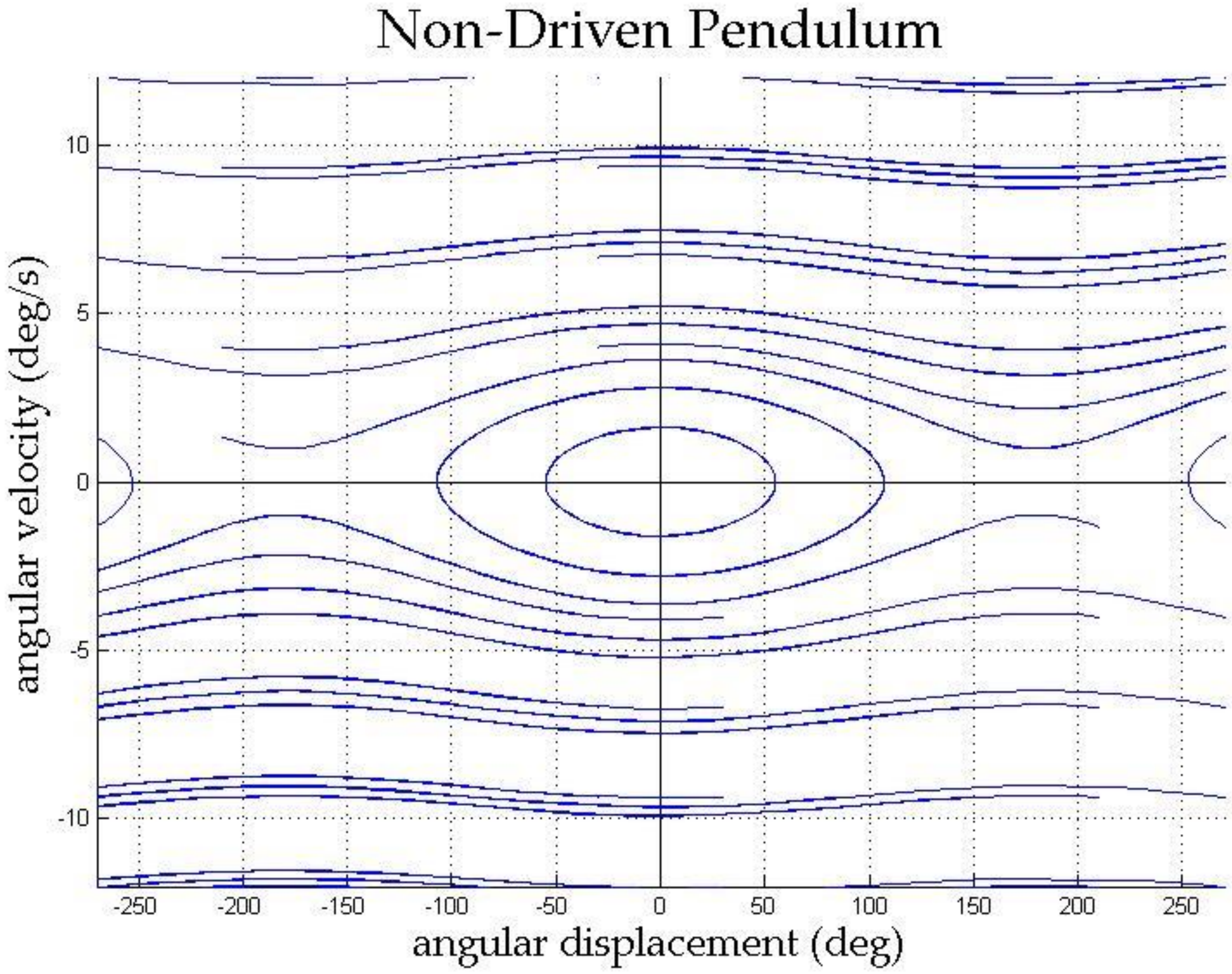
$\gamma = 1.6x10^{-5}$

stable

unstable

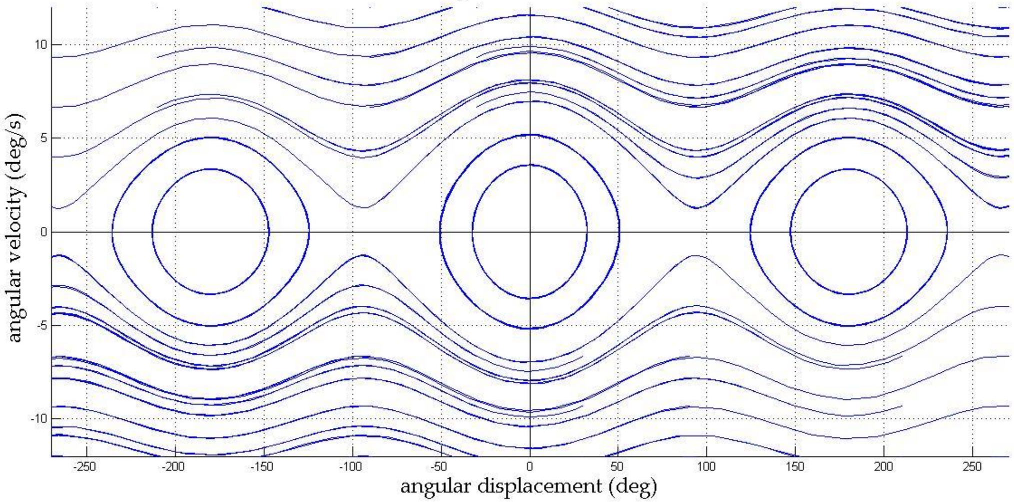
AVERAGING & EFFECTIVE POTENTIAL U_{EFF}

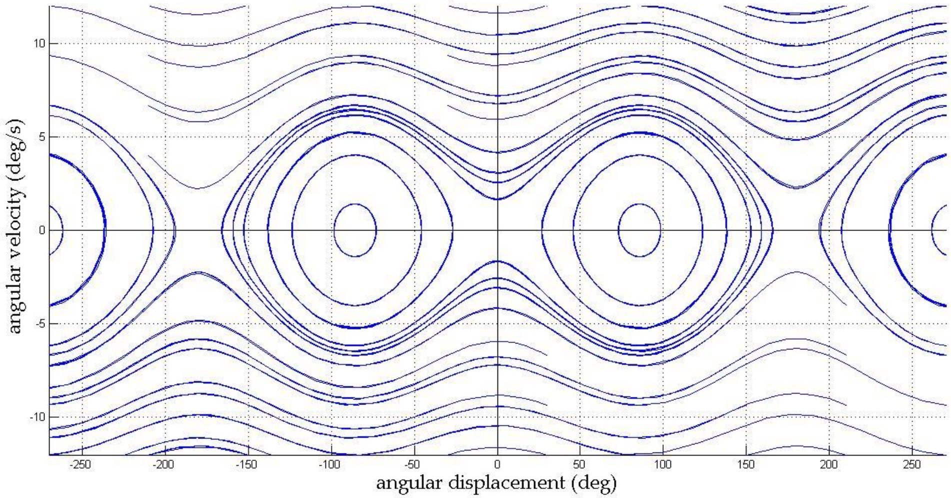
PHASE PORTRAIT



AVERAGING & EFFECTIVE POTENTIAL U_{EFF}

PHASE PORTRAIT

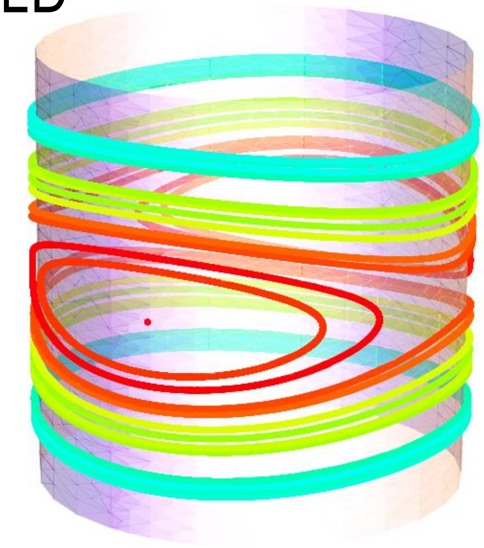

Vertically Driven Pendulum


Horizontally Driven Pendulum

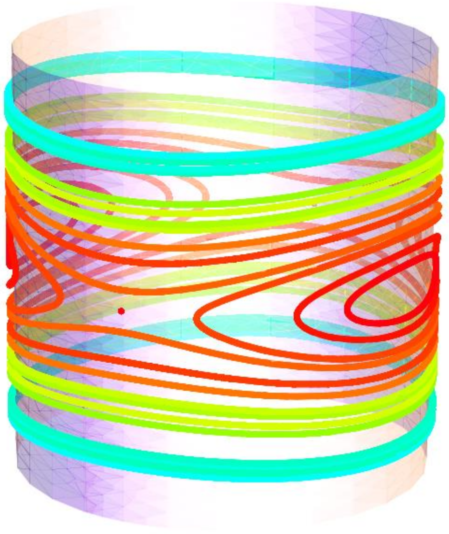
AVERAGING & EFFECTIVE POTENTIAL U_{EFF}

DYNAMIC MANIFOLD

Non-Driven Pendulum



Vertically-Driven



Horizontally-Driven

EXPERIMENTAL

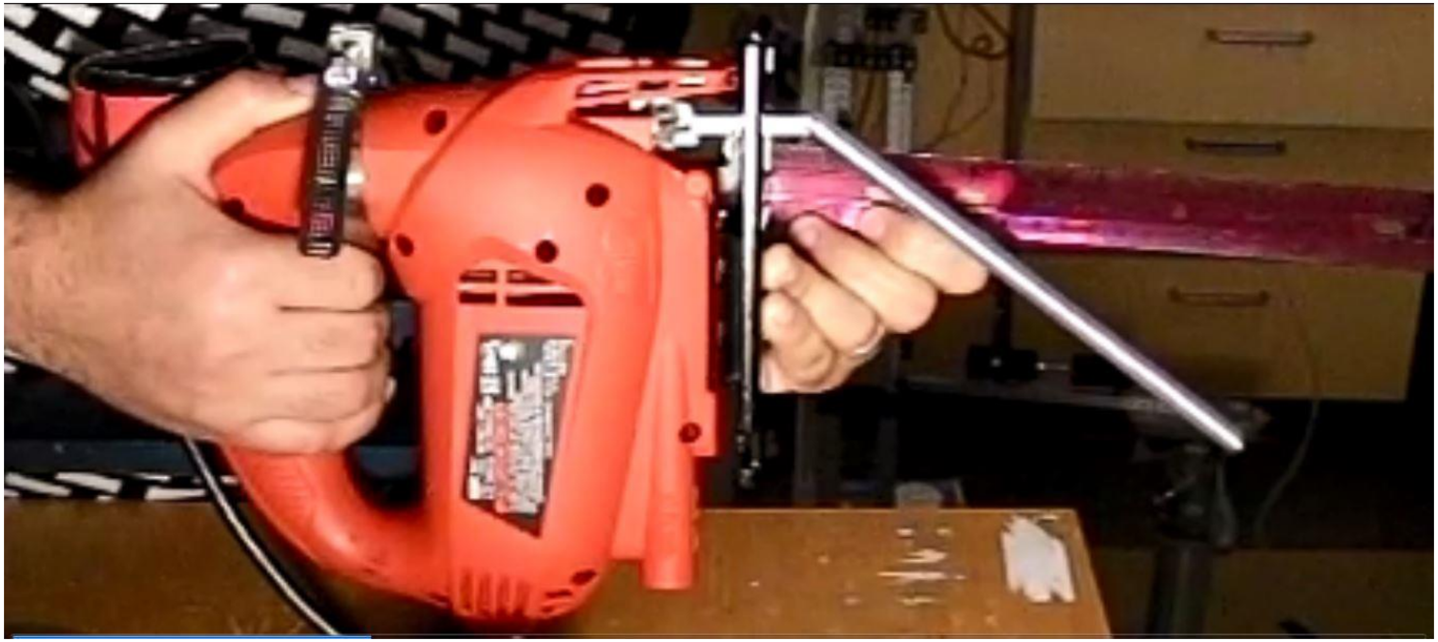
OVERVIEW

Tools

- Black & Decker JS515 Jigsaw
- High Speed 6000 Frames per Second Camera

Procedure

- Frame by frame analysis
- On-screen measurement



EXPERIMENTAL

FRAME BY FRAME ANALYSIS

Measurement (driving)	Theoretical	Experimental
Vertical Stability Angle	π (straight up)	π
Horizontal Stability Angle	1.477 radians	~1.405 radians
Diagonal Stability Angle	0 (straight down)	-
Vertical Frequency	314.16 dps	300.3 dps
Horizontal Frequency	314.16 dps	453 dps

QUESTIONS?