W – FLOWS ON WEYL MANIFOLDS AND GAUSSIAN THERMOSTATS

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Abstract. We introduce W-flows, by modifying the geodesic flow on a Weyl manifold, and show that they coincide with the isokinetic dynamics. We establish some connections between negative curvature of the Weyl structure and the hyperbolicity of W-flows, generalizing in dimension 2 the classical result of Anosov on Riemannian geodesic flows. In higher dimensions we establish only weaker hyperbolic properties. We extend the theory to billiard W-flows and introduce the Weyl counterparts of Sinai billiards. We obtain that the isokinetic Lorentz gas with the constant external field E and scatterers of radius r, studied by Chernov, Eyink, Lebowitz and Sinai in [Ch-E-L-S], is uniformly hyperbolic, if only $r|E| < 1$, and this condition is sharp. **To appear in Journal de Mathematique Purees et Appliques**

§1. Introduction.

A Weyl structure on a manifold generalizes a Riemannian metric. It is a torsion free connection with the parallel transport preserving a given conformal class of metrics. A presentation of Weyl manifolds in the modern language was given by Folland, [F].

Fixing a Riemannian metric in this class we introduce the W-flow on the unit tangent bundle whose orbits are geodesics of the Weyl connection equipped with unit tangent vectors. Such flows turn out to be identical with Gaussian thermostats, or isokinetic dynamics, introduced by Hoover [H]. Isokinetic dynamics provides useful models in nonequilibrium statistical mechanics, discussed in the papers of Gallavotti and Ruelle, [G],[R], [G-R].

With a fixed Riemannian metric the Weyl structures $\widehat{\nabla}$ are parametrized by 1-forms on the manifold or equivalently by vector fields E

$$\widehat{\nabla}_X Y = \nabla_X Y + \langle X, E\rangle Y + \langle Y, E\rangle X - \langle X, Y\rangle E,$$

where X, Y denote arbitrary vector fields, $\langle , \rangle$ denotes the chosen Riemannian metric and ∇ is its Levi-Civita connection. If we multiply the metric by a function e^{-2U}, then the vector field E is replaced by $e^{2U}(E + \nabla U)$. Hence, if the vector field E has a potential i.e., $E = -\nabla U$ then the Weyl structure coincides with the Levi-Civita connection of the rescaled metric. This leads to the local variational formulation of the isokinetic dynamics obtained by Choquard, [Cho]. Since the geodesic flows are Hamiltonian when considered on the

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cotangent bundle we get the Hamiltonian formulation of the Gaussian thermostat obtained by Dettmann and Morriss, [D-M]. This also connects us with the conformally symplectic formulation of the isokinetic dynamics in [W-L]. Thus the point of view presented in this paper generalizes the previous interpretations of isokinetic dynamics.

Using the Jacobi equations for a geodesic flow we obtain the linearized equations for W-flows in terms of the curvature tensor of the Weyl connection. The Weyl curvature operator has a symmetric component (distance curvature) which is a multiple of the identity. The antisymmetric part of the curvature leads to sectional curvatures in planar directions, which can be defined only with respect to some fixed metric. However the sign of the sectional curvatures has absolute character. Sectional curvatures for the Weyl structures defined by a constant vector field on a flat two dimensional torus are zero but it is not the case in higher dimensions. We obtain negative sectional curvature for any plane which does not contain the vector field E.

We investigate the Anosov property for the W-flows under the assumption that the sectional curvatures are negative. In dimension 2 we obtain the counterpart of the Anosov theorem for geodesic flows, negative sectional curvatures do imply that the W-flow is Anosov. In dimension $n \geq 3$ we obtain only two invariant distributions of n dimensional tangent subspaces, with one of them repelling and the other contracting most of other distributions, i.e., we obtain the dominant splitting for the flow (cf. [$\tilde{\text{M}}$]). We establish additionally that volumes are expanded on one invariant distribution and contracted on the other. We were unable to obtain any further hyperbolic properties for the dynamics. We conjecture that there are 3 dimensional manifolds and vector fields E such that the sectional curvatures of the corresponding Weyl structure are negative but the W-flow is not Anosov.

An interesting example is furnished by negatively curved surfaces with a harmonic 1-form representing a nonzero cohomology class. The corresponding vector field gives no contribution into the sectional curvatures of the Weyl connection so that we can multiply the field by a large scalar and the resulting flows are always Anosov. These systems were studied by Bonetto, Gentile and Mastropietro, [B-G-M], with the goal of establishing the singular nature of the SRB measure for small fields. We obtain it for fields of arbitrary strength. It would be interesting to find the asymptotic behavior of the SRB measures for these flows. We conjecture that there is a limit measure supported on the integral curves of the field.

We apply these ideas to the periodic Lorentz gas driven by an external field with the addition of the Gaussian thermostat, the system studied numerically by Moran and Hoover, [M-H], and analyzed rigorously by Chernov, Eyink, Lebowitz and Sinai, [Ch-E-L-S], in the case of a weak field.

It turns out that in the plane the conformal mapping by e^z transforms the trajectories of the system into straight segments and hence the character of the system depends on the convexity of the transformed scatterers. Small disks are mapped by e^z into convex domains but beyond radius 1 they develop a dip on one side. This dip is responsible for the stable periodic orbits observed numerically in [M-H] for sufficiently large fields. We obtain explicit and sharp conditions for the uniform hyperbolicity of the system, which paves the way for extending the analysis in [Ch-E-L-S] beyond the case of very small fields.

In higher dimensions the convexity of the scatterers with respect to the Weyl structure is apparently not sufficient to guarantee the complete hyperbolicity. More work is required to resolve this issue.

§2. Gaussian thermostats and W – flows.

Let us consider a compact n-dimensional Riemannian manifold M and its tangent bundle TM. The metric g will be also denoted by $\langle\cdot,\cdot\rangle$. For a smooth vector field E on M we introduce the flow on TM defined by the equations

$$\frac{dq}{dt} = v, \frac{Dv}{dt} = E,$$

where $\frac{D}{dt}$ denotes the covariant derivative ∇_v. We further impose the preservation of the kinetic energy $H = \frac{1}{2}v^2$ via the Gauss least constraint principle, [H]. We obtain *the Gaussian thermostat* or *the isokinetic dynamics.*

$$\frac{dq}{dt} = v, \frac{Dv}{dt} = E - \frac{\langle E, v\rangle}{v^2}v.$$

We fix the value of the kinetic energy, $v^2 = c$, so that the phase space becomes the sphere bundle. By rescaling time we can assume that $v^2 = 1$ and then the vector field E is multiplied by the factor $\frac{1}{c}$. We will be considering Gaussian thermostats only in this restricted phase space, i.e, in the unit tangent bundle SM of M.

Let φ be the 1-form associated with the vector field E, i.e., $\varphi(\cdot) = \langle E,\cdot\rangle$. Together with the Riemannian metric it defines a Weyl structure on M, which is a linear torsion free connection $\widehat{\nabla}$ given by the formula (cf. [F])

$$\widehat{\nabla}_X Y = \nabla_X Y + \varphi(Y)X + \varphi(X)Y - \langle X, Y\rangle E,$$

for any vector fields X, Y on M (by ∇ we denote the Riemannian connection). The Weyl structure is usually introduced on the basis of the conformal class of g rather than g itself, but in our study we fix the Riemannian metric, which plays the role of a physical space. If we change the metric g by the factor e^{-2U} to $\widetilde{g} = e^{-2U}g$, then the 1-form $\varphi(\cdot) = \langle E,\cdot\rangle$ is replaced by $\widetilde{\varphi} = \varphi + dU$. Hence, if the vector field E has a potential i.e., $E = -\nabla U$ then the Weyl structure coincides with the Levi-Civita connection of the rescaled metric $\widetilde{g}$.

The defining property of the Weyl structure is (cf. [F])

$$\widehat{\nabla}_X g = -2\varphi(X)g. \tag{2.1}$$

We consider the geodesics of the Weyl connection. They are given by the equations in TM

$$\frac{dq}{ds} = w, \quad \frac{\widehat{D}w}{ds} = 0, \tag{2.2}$$

where $\frac{\widehat{D}}{ds}$ denotes the covariant derivative $\widehat{\nabla}_w$. These equations provide geodesics with distinguished parameter s defined uniquely up to scale. It follows from (2.1) and (2.2) that $\frac{d|w|}{ds} = -\varphi(w)|w|$. Assuming that at the initial point $q(0)$ we have $|w| = 1$ we obtain

$$|w| = e^{-\int_{q(0)}^{q(s)} \varphi}.$$

This formula shows that unless the form φ is exact we should not expect the geodesic flow in TM of a Weyl connection to preserve any sphere bundle. We introduce the flow

$$\Phi^t : SM \to SM,$$

called the *W-flow*, by parametrizing the geodesics of the Weyl connection with the arc length of g. In other words the projection of a trajectory of Φ^t to M is a geodesic of the Weyl connection, t is the arc length parameter defined by the metric g and the trajectory itself is the natural lift of the geodesic to SM.

Theorem 2.1. *The W-flow on SM coincides with the Gaussian thermostat on SM*

$$\frac{dq}{dt} = v, \quad \frac{Dv}{dt} = E - \langle E, v\rangle v. \tag{2.3}$$

Proof. Let us consider a geodesic $q(s), w(s) = \frac{dq}{ds}$, of the Weyl connection. The parameter s is related to the arc length parameter t by $\frac{dt}{ds} = |w|$. We obtain

$$\frac{dq}{dt} = v = \frac{w}{|w|}, \quad \nabla_v v = \frac{1}{w^2}\nabla_w w + \varphi(v)v = E - \varphi(v)v,$$

It shows that the reparametrized geodesic satisfies (2.3) □

The W-flow (just like the geodesic flow) is *reversible* in the following sense. If we denote by $\Psi : SM \to SM$ the involution $\Psi(q, v) = (q, -v)$ then we have

$$\Phi^t\Psi = \Psi\Phi^{-t}. \tag{2.4}$$

In the case of a potential vector field $E = -\nabla U$, the Weyl connection is the Levi-Civita connection of the metric $\widetilde{g} = e^{-2U}g$ and hence up to reparametrization the trajectories of the W-flow are the geodesics of this metric.

The interesting case is that of a field with only local potential, i.e., when φ is closed but not exact. In such a situation the metric $\widetilde{g} = e^{-2U}g$ is only local. Note that the Levi-Civita connection and the geodesics are not changed, if we multiply the Riemannian metric by a constant factor, so that although the metric is local the connection and the geodesics are globally defined. To obtain the W-flow we parametrize these geodesics by the arc length with respect to the background metric g.

Example 2.2.

Let $\mathbb{T}^2$ be the flat torus with coordinates $(x, y) \in \mathbb{R}^2$ and $E = (a, 0)$ be the constant vector field on $\mathbb{T}^2$ then the equations of the W-flow

$$\ddot{x} = a\dot{y}^2, \quad \ddot{y} = -a\dot{x}\dot{y},$$

can be integrated and we obtain as trajectories translations of the curve

$$ax = -\ln \cos ay$$

or the horizontal lines. Assuming that E has irrational direction on $\mathbb{T}^2$ we obtain the following global phase portrait for the W-flow. In the unit tangent bundle $S\mathbb{T}^2 = \mathbb{T}^3$ we have two invariant tori A and R with the minimal quasiperiodic motions, A contains the unit vectors in the direction of E and it is a global attractor and R contains the unit vectors opposite to E and it is a global repellor. We will establish later on (proof of Proposition 5.9) that these invariant submanifolds are normally hyperbolic so that the phase portrait is preserved under perturbations.

This example reveals the major departure from geodesic flows and Hamiltonian dynamics. W-flows may contract phase volume and they may have no absolutely continuous invariant measure.

Example 2.3.

More generally, let N be an arbitrary Riemannian manifold and let $M = N \times S^1$ with the product metric. Let E be the vector field tangent to S^1 of constant magnitude $|E|$. For a velocity vector $v \in SM$ let $v = v_1 + v_0$ be the splitting into the component v_1 parallel to N and the component v_0 parallel to S^1. We can immediately calculate how v_0 depends on time. Assuming that at $t = 0$, $v_0 = 0$ we get $v_0 = \tanh |E|t$. Moreover the projection of a trajectory of the W-flow onto N is an open segment of a geodesic of length $\frac{\pi}{|E|}$, with the exception of the trajectories along S^1 which project to a single point. Just as in Example 2.2 we obtain the global attractor and the global repellor. The attractor and repellor are also normally hyperbolic, if only $|E|$ is sufficiently large.

Example 2.4.

It was calculated in [W1] that the rate of dilation of the phase volume in SM (the divergence of the velocity vector field) for isokinetic dynamics (a W-flow in our new terminology) is equal to $-(n-1)\langle E, v\rangle$ (n is the dimension of M). We have

$$\frac{d}{dt}\langle E, v\rangle = \langle \nabla_v E, v\rangle + E^2 - \langle E, v\rangle^2,$$

Assuming that E does not vanish and multiplying it by a sufficiently large positive scalar we obtain the flow which crosses the submanifold $\langle E, v\rangle = 0$ in one direction only. Moreover on one side of this submanifold, where $\langle E, v\rangle < 0$, we have volume expansion and on the other side, where $\langle E, v\rangle > 0$, volume is contracted. Hence in particular such flows cannot have absolutely continuous invariant measures.

Let E be a nonvanishing Killing vector field on a Riemannian manifold M. Then $\langle \nabla_X E, Y\rangle$ is an antisymmetric function of X, Y. Thus we obtain

$$\frac{d}{dt}\langle E, v\rangle = E^2 - \langle E, v\rangle^2 \geq 0,$$

and the directed crossing of the submanifold $\langle E, v\rangle$ follows, independently of the magnitude of E.

§3. Jacobi equations and curvature.

We will obtain the linearized equations of the W-flow from the Jacobi equations for the geodesic flow of a torsion free connection. Let us consider a family of geodesics of the Weyl connection

$$q(s,u), w(s,u) = \frac{dq}{ds}, |u| < \epsilon.$$

We introduce the Jacobi field

$$\xi = \frac{dq}{du} \quad \text{and} \quad \widehat{\eta} = \widehat{\nabla}_\xi w.$$

Note that $\widehat{\nabla}_\xi w = \widehat{\nabla}_w \xi$ since our connection $\widehat{\nabla}$ has zero torsion and the fields w and ξ commute. We have the Jacobi equations

$$\frac{\widehat{D}\xi}{ds} = \widehat{\eta}, \quad \frac{\widehat{D}\widehat{\eta}}{ds} = -\widehat{R}(\xi, w)w, \tag{3.1}$$

where for any tangent vector fields X, Y,

$$\widehat{R}(X,Y) = \widehat{\nabla}_X\widehat{\nabla}_Y - \widehat{\nabla}_Y\widehat{\nabla}_X - \widehat{\nabla}_{[X,Y]}$$

is the curvature tensor of the Weyl connection.

Let us split the vector field $\widehat{\eta} = \widehat{\eta}_0 + \widehat{\eta}_1$ into the component $\widehat{\eta}_1$ orthogonal to w and the component $\widehat{\eta}_0$ parallel to w. The equations (3.1) can be split accordingly

$$\frac{\widehat{D}\widehat{\eta}_1}{ds} = -\widehat{R}_a(\xi, w)w, \quad \frac{\widehat{D}\widehat{\eta}_0}{ds} = -\widehat{R}_s(\xi, w)w, \tag{3.2}$$

where $\widehat{R}_a(X,Y)$ is the antisymmetric and $\widehat{R}_s(X,Y)$ the symmetric part of the Weyl curvature operator $\widehat{R}(X,Y) = \widehat{R}_a(X,Y) + \widehat{R}_s(X,Y)$ ($\widehat{R}_s$ is called the distance curvature and $\widehat{R}_a$ the direction curvature, cf. [F]). To obtain (3.2) let us observe that since $\widehat{\eta}_1$ is parallel to w also $\frac{\widehat{D}\widehat{\eta}_1}{ds}$ is parallel to w. Further we calculate

$$\langle\frac{\widehat{D}\widehat{\eta}_0}{ds}, w\rangle = \frac{d}{ds}\langle\widehat{\eta}_0, w\rangle + 2\varphi(w)\langle\widehat{\eta}_0, w\rangle = 0.$$

Now (3.2) follows from the general fact that $\widehat{R}_a(w,\xi)w$ is orthogonal to w and from $\widehat{R}_s(w,\xi)w$ being parallel to w which is a consequence of the fact that the distance curvature is a multiple of identity (cf. [F] and Proposition 3.2 below).

We will now calculate straightforwardly the curvature tensor of the Weyl connection.

Proposition 3.2.

$$\begin{aligned}\widehat{R}_a(X,Y)Z =& R(X,Y)Z + \langle Z,E\rangle(\langle Y,E\rangle X - \langle X,E\rangle Y)+\\ &(\langle Z,Y\rangle\langle X,E\rangle - \langle Z,X\rangle\langle Y,E\rangle)E + E^2(\langle Z,X\rangle Y - \langle Z,Y\rangle X)+\\ &\langle Z,\nabla_X E\rangle Y - \langle Z,\nabla_Y E\rangle X + \langle Z,X\rangle\nabla_Y E - \langle Z,Y\rangle\nabla_X E.\\ \widehat{R}_s(X,Y)Z =& (\langle Y,\nabla_X E\rangle - \langle X,\nabla_Y E\rangle)Z = -d\varphi(X,Y)Z\end{aligned}$$

□

Together with the family of Weyl geodesics let us consider the respective family of trajectories of the W-flow

$$q(t,u), v(t,u) = \frac{dq}{dt}, |u| < \epsilon.$$

The Jacobi field is again $\xi = \frac{dq}{du}$. Letting $\eta = \nabla_\xi v = \nabla_v \xi$ we can consider (ξ,η) as coordinates in the tangent space of SM, which is described by $\langle v,\eta\rangle = 0$. We introduce

$$\chi = \eta + \langle E,v\rangle\xi - \langle \xi,v\rangle E$$

and use (ξ,χ) as linear coordinates in the tangent bundle of SM. Note that $\langle v,\eta\rangle = 0$ is equivalent to $\langle v,\chi\rangle = 0$ and in these new coordinates the velocity vector field of the W-flow (2.3) is simply $(v,0)$. The meaning of χ is revealed in the following proposition describing the linearization of a W-flow.

Proposition 3.3. *We have that* $\chi = \frac{\widehat{\eta}_1}{|w|}$, *i.e.,* χ *is the component of* $\widehat{\nabla}_\xi v$ *orthogonal to* v, *and*

$$\frac{\widehat{D}\xi}{dt} = \chi + \varphi(\xi)v, \quad \frac{\widehat{D}\chi}{dt} = -\widehat{R}_a(\xi,v)v + \varphi(v)\chi. \tag{3.3}$$

Proof. Using the equations (3.2) we obtain

$$\frac{\widehat{D}\xi}{dt} = \widehat{\nabla}_v\xi = \frac{\widehat{\eta}}{|w|} = \frac{\widehat{\eta}_0}{|w|} + \frac{\widehat{\eta}_1}{|w|}.$$

On the other hand

$$\widehat{\nabla}_v\xi = \nabla_v\xi + \varphi(v)\xi + \varphi(\xi)v - \langle v,\xi\rangle E = \chi + \varphi(\xi)v,$$

which leads to $\chi = \frac{\widehat{\eta}_1}{|w|}$ and the first equation of (3.3). Using (3.2) we get

$$\frac{\widehat{D}\chi}{dt} = \widehat{\nabla}_v\chi = \frac{1}{w^2}\widehat{\nabla}_w\widehat{\eta}_1 + \frac{d}{dt}(\frac{1}{|w|})\widehat{\eta}_1 = -\widehat{R}_a(\xi,v)v + \varphi(v)\chi.$$

□

The Weyl parallel transport along a path is a conformal linear mapping and the coefficient of dilation is equal to $e^{-\int\varphi}$. We choose an orthonormal frame $v, e_1, \dots, e_{n-1}$ in an initial tangent space $T_{q_0}M$ and parallel transport it along a trajectory of our W-flow in the direction $v \in SM$. We obtain the orthogonal frames which we normalize by the coefficient $e^{\int\varphi}$ and denote them by $v(t), e_1(t), \dots, e_{n-1}(t)$. Let $(\xi_0, \xi_1, \dots, \xi_{n-1})$ and $(0, \chi_1, \dots, \chi_{n-1})$ be the components of ξ and χ respectively in these frames. Let further $\widetilde{\xi} = (\xi_1, \dots, \xi_{n-1}) \in \mathbb{R}^{n-1}$ and $\widetilde{\chi} = (\chi_1, \dots, \chi_{n-1}) \in \mathbb{R}^{n-1}$. The equations (3.3) will read then

$$\frac{d\xi_0}{dt} = \varphi(\xi - \langle \xi, v\rangle v), \quad \frac{d\widetilde{\xi}}{dt} = -\varphi(v)\widetilde{\xi} + \widetilde{\chi}, \quad \frac{d\widetilde{\chi}}{dt} = -\widehat{R}_a(\xi, v)v, \tag{3.4}$$

where the vector $\widehat{R}_a(\xi, v)v$, being orthogonal to v, is considered as an element in $\mathbb{R}^{n-1}$ by the expansion in the basis $e_1(t), \dots, e_{n-1}(t)$. Note that $\varphi(\xi - \langle \xi, v\rangle v)$ and $\widehat{R}_a(\xi, v)v$ depend only on $\widetilde{\xi} \in \mathbb{R}^{n-1}$. The operator $\widetilde{\xi} \to \widehat{R}_a(\xi, v)v \in \mathbb{R}^{n-1}$, in contrast to the Riemannian case, is not in general symmetric.

We introduce the sectional curvatures $\widehat{K}(\Pi)$ of the Weyl structure in the direction of a plane Π as

$$\widehat{K}(\Pi) = \langle \widehat{R}_a(X, Y)Y, X\rangle,$$

for any orthonormal basis $\{X, Y\}$ of Π. The sectional curvatures depend on the choice of a Riemannian metric g in the conformal class and not on the Weyl structure alone. However the sign of sectional curvatures is well defined.

Denoting by $K(\Pi)$ the Riemannian sectional curvature in the direction of Π and by E_Π the orthogonal projection of E on Π we obtain from Proposition 3.2

$$\widehat{K}(\Pi) = K(\Pi) - \left(E^2 - E_\Pi^2\right) - \langle \nabla_X E, X\rangle - \langle \nabla_Y E, Y\rangle. \tag{3.5}$$

If M is 2-dimensional the terms in the brackets drop out and $\langle \nabla_X E, X\rangle + \langle \nabla_Y E, Y\rangle$ is the divergence of the vector field E. By the theorem of Gauduchon, [Ga], we can always multiply the metric by a positive function and make the new vector field E divergence free (in dimension 2 this theorem follows from the theory of harmonic forms). Such a special choice of the metric is called the Gauduchon gauge. We obtain that in dimension 2 the curvature of a Weyl structure with respect to the Gauduchon gauge is equal to the Gaussian curvature.

In general let us denote by $div_\Pi E = \langle \nabla_X E, X\rangle + \langle \nabla_Y E, Y\rangle$ the partial divergence of the vector field E, i.e., the exponential rate of growth of the area in the direction Π under the flow of the vector field E. Then we can rewrite the formula (3.5) as

$$\widehat{K}(\Pi) = K(\Pi) - \left(E^2 - E_\Pi^2\right) - div_\Pi E. \tag{3.6}$$

Let us inspect the Examples 2.2 and 2.3 for the presence of negative sectional curvatures. Taking a constant vector field E on a flat torus in dimension 2 we obtain the Weyl structure with zero curvature tensor. Moreover by the previous observation, any Weyl structure on the 2-dimensional torus with nonpositive sectional curvature must have zero sectional curvature. In dimension 3 and higher the Weyl structure has negative sectional curvatures everywhere except for the planes containing the vector E, in contrast to the Riemannian case.

More generally in the Example 2.3 for any Riemannian metric on N with $dimN \geq 2$, if the magnitude of the field E exceeds the maximal positive sectional curvature of N then the Weyl sectional curvatures of M are negative except for the planes containing E on which the Weyl sectional curvature is zero.

Using the normally hyperbolic invariant submanifolds described in Examples 2.2, 2.3 we will arrive at the conclusion (Proposition 5.9) that small perturbations of these Weyl manifolds cannot have strictly negative sectional curvatures everywhere.

Clearly one can obtain manifolds with negative Weyl sectional curvatures by starting with the Riemannian metric of negative sectional curvature and allowing sufficiently small fields E. We do not know any nonperturbative examples.

Problem 3.4. *Construct a Weyl structure with negative sectional curvatures on a manifold which does not carry a Riemannian metric of negative sectional curvature.*

4. Monotonicity, $\mathcal{J}$–algebra of Potapov and dominated splittings.

To study the hyperbolic properties of the W-flows we will be using general criteria developed in [W2]. They are a combination of the ideas of Lewowicz, [L] and our, [W3], [W4], [W5], with the $\mathcal{J}$-algebra of Potapov, [P1],[P2],[P3]. For the convenience of the reader we will formulate them briefly. The detailed exposition can be found in [W2]. Let us consider an n-dimensional linear space V with a chosen pseudo Euclidean structure, i.e., a nondegenerate quadratic form $\mathcal{J}$ with the positive index of inertia equal to p and the negative index of inertia equal to q, $p+q=n, p \geq 1, q \geq 1$. Depending on the sign of $\mathcal{J}$ we have positive and negative vectors. We denote by $\mathcal{C}_\pm = \{v \in V | \pm\mathcal{J}(v) > 0\} \cup \{0\}$ the open cones of, respectively, positive and negative vectors, and by $\mathcal{C}_0$ their common boundary, $\mathcal{C}_0 = \{v \in V | \mathcal{J}(v) = 0\}$.

A linear subspace is called positive (negative), if it is contained in $\mathcal{C}_+$ ($\mathcal{C}_-$). The maximal dimension of a positive subspace is p and of a negative subspace q.

Definition 4.1. *A linear nonsingular operator $S : V \to V$ is*

1. *$\mathcal{J}$–separated, if $S\mathcal{C}_+ \subset \mathcal{C}_+$,*
2. *strictly $\mathcal{J}$–separated, if $S(\mathcal{C}_0 \cup \mathcal{C}_+) \subset \mathcal{C}_+$,*
3. *$\mathcal{J}$–monotone, if $\mathcal{J}(Sv) \geq \mathcal{J}(v)$ for every $v \in V$,*
4. *strictly $\mathcal{J}$–monotone, if $\mathcal{J}(Sv) > \mathcal{J}(v)$ for every $v \in V, v \neq 0$,*
5. *$\mathcal{J}$–isometry, if $\mathcal{J}(Sv) = \mathcal{J}(v)$ for every $v \in V$,*
6. *$\mathcal{J}$–symmetric if $\mathcal{J}(S\cdot,\cdot)$ is a symmetric bilinear form on V.*

There are natural infinitesimal versions of properties 1 – 5 for one parameter subgroups of linear operators (elements of the Lie algebra).

Potapov, [P1],[P2],[P3], obtained fundamental results about the structure and properties of $\mathcal{J}$–monotone operators, which he called $\mathcal{J}$–noncontractive. He developed the theory for a pseudo unitary form $\mathcal{J}$ rather than pseudo Euclidean, but in our applications we need only the real version.

Every $\mathcal{J}$–separated operator S has a unique representation $S = RU$ where U is a $\mathcal{J}$–isometry and R is $\mathcal{J}$–symmetric and has only positive eigenvalues. The operator R is equal to $(SJS^*J)^{\frac{1}{2}}$ and it is called the modulus of S.

Let us denote the eigenvalues of the modulus of S by

$$r_p^+ \geq \cdots \geq r_2^+ \geq r_1^+ \geq r_1^- \geq r_2^- \geq \cdots \geq r_q^- > 0.$$

S is strictly $\mathcal{J}$–separated iff $r_1^+ > r_1^-$. It is strictly $\mathcal{J}$–monotone iff $r_1^+ > 1 > r_1^-$.

The eigenvalues of the modulus can be interpreted as pseudo Euclidean singular values. For a d–dimensional positive linear subspace $L \subset \mathcal{C}_+$ the restriction of $\mathcal{J}$ is a scalar product. If S is $\mathcal{J}$–separated then $SL \subset \mathcal{C}_+$ is also a positive subspace with the scalar product furnished by $\mathcal{J}$. Hence the subspaces L and SL are equipped with (unoriented) volume elements, which we call the $\mathcal{J}$–volumes. Let $\alpha_d(L;S)$ be the coefficient of the $\mathcal{J}$–volume expansion of the linear map S restricted to L. Further let $\sigma_d(S)$ be the infimum of $\alpha_d(L;S)$ over all positive d–dimensional subspaces L. We have

Proposition 4.1[W2]**.** $\sigma_d(S) = r_1^+ r_2^+ \ldots r_d^+$.

Similarly we can obtain the other eigenvalues of the modulus by considering the negative subspaces. Note that if S is $\mathcal{J}$-separated then S^{-1} is $(-\mathcal{J})$-separated.

Let us consider the manifold $\mathcal{L}_+$ of all p-dimensional subspaces contained in $\mathcal{C}_+$. It can be identified with the manifold of $q \times p$ matrices T such that $T^*T < I_p$. This manifold carries the Riemannian metric of the symmetric space [Py],

$$(dT)^2 = tr\left((I_p - T^*T)^{-1} dT^* (I_q - TT^*)^{-1} dT\right).$$

A $\mathcal{J}$–separated operator S maps naturally $\mathcal{L}_+$ into itself. This mapping is a contraction. More precisely we have

Theorem 4.2[W2]**.** *For a $\mathcal{J}$–separated operator S and $T_1, T_2 \in \mathcal{L}_+$*

$$dist(ST_1, ST_2) \leq \frac{r_1^-}{r_1^+} dist(T_1, T_2).$$

Let us move the discussion to a compact manifold M of dimension n and a diffeomorphism $\Phi : M^n \to M^n$. We assume that the manifold is equipped with a continuous pseudo Riemannian metric of type $(p,q), p+q = n$, which we will denote again by $\mathcal{J}$.

We obtain the field of positive and negative cones $\mathcal{C}_\pm(x), x \in M$. A diffeomorphism Φ is called $\mathcal{J}$–separated if $D\Phi\mathcal{C}_+(x) \subset \mathcal{C}_+(\Phi x)$. Similarly we define diffeomorphisms which are strictly $\mathcal{J}$–separated, $\mathcal{J}$–monotone and strictly $\mathcal{J}$–monotone.

Theorem 4.3[W2]**.** *If a diffeomorphism Φ is strictly $\mathcal{J}$–separated then it has a dominated splitting, i.e., it has a continuous invariant splitting of the tangent bundle*

$$T_xM = \mathcal{E}^-(x) \oplus \mathcal{E}^+(x),$$

such that for some $0 < \lambda < 1, c > 0$, and all $k \geq 1$

$$\|D\Phi^k{}_{|\mathcal{E}^-}\| \; \|D\Phi^{-k}{}_{|\mathcal{E}^+}\| < c\lambda^k.$$

Proof. At every point $x \in M$ we have the manifold of positive subspaces $\mathcal{L}_+(x)$. We will use the contraction property of the action of $D\Phi$ on $\mathcal{L}_+(x)$. The diameter of $D\Phi\mathcal{L}_+(x)$ is finite because $D\Phi$ is strictly $\mathcal{J}$–separated, and hence it is uniformly bounded by a constant c_0. It follows from Theorem 4.2 that the diameter of $D\Phi^k\mathcal{L}_+(\Phi^{-k}x)$ does not exceed $c_0\prod_{i=1}^{k-1}\frac{r_1^-(\Phi^{-i}x)}{r_1^+(\Phi^{-i}x)}$. Since the subsets

$$D\Phi^{k+1}\mathcal{L}_+(\Phi^{-k-1}x) \subset D\Phi^k\mathcal{L}_+(\Phi^{-k}x)$$

are nested and their diameter decays exponentially we get a unique point of intersection

$$\mathcal{E}^+(x) = \bigcap_{k=1}^{\infty} D\Phi^k\mathcal{L}_+(\Phi^{-k}x). \tag{4.1}$$

It is clear that the subspace depends continuously on x. To obtain the other subspace of the splitting we need to observe that Φ^{-1} is strictly $(-\mathcal{J})$–separated and repeat the above argument.

Finally we get the separation property with $\lambda = \sup_x \frac{r_1^-(x)}{r_1^+(x)}$.

□

As a corollary we obtain the theorem of Lewowicz, [L],

Theorem 4.5. *If a diffeomorphism is strictly $\mathcal{J}$–monotone then it is Anosov.*

Proof. By Theorem 4.3 we obtain the continuous invariant splitting. On the unstable subspace the $\mathcal{J}$–form increases exponentially and it can be estimated by any norm.

□

For an ergodic invariant measure ν we can obtain the estimates of the Lyapunov exponents in terms of the pseudo Riemannian singular values of the $\mathcal{J}$–separated $D\Phi$, i.e., the eigenvalues of the modulus of $D\Phi$. Having in mind the applications to billiard W-flows we allow piecewise differentiable mappings.

Theorem 4.6[W2]. *For a piecewise differentiable $\mathcal{J}$–separated Φ and an ergodic invariant measure ν, if $\log\|D\Phi^{\pm1}\|$ are integrable functions then the Lyapunov exponents*

$$\mu_{-p} \le \mu_{-p-1} \le \cdots \le \mu_{-1} \le \mu_1 \le \mu_2 \le \cdots \le \mu_q,$$

satisfy

$$\mu_1 + \cdots + \mu_{k_1} \ge \sum_{i=1}^{k_1}\int \log r_i^+ d\nu \quad \text{and} \quad \sum_{i=1}^{k_2}\int \log r_i^- d\nu \ge \mu_{-1} + \cdots + \mu_{-k_2},$$

for any $k_1 \le p, k_2 \le q$, where the coefficients $0 < r_q^-(x) \le \cdots \le r_1^-(x) \le r_1^+(x) \le \cdots \le r_p^+(x)$ are the singular values of the $\mathcal{J}$–separated operators $D_x\Phi$.

To develop the theory of $\mathcal{J}$–separated flows $\Phi^t : M^n \to M^n, t \in \mathbb{R}$, we need the velocity vector field $F(x)$ of the flow, $F(x) = \frac{d}{dt}\Phi^t(x)_{|t=0}$, to be nonzero everywhere and the form

$\mathcal{J}$ to be degenerate. More precisely the positive index of inertia of $\mathcal{J}$ is equal to $p \geq 1$, the negative index of inertia is equal to $q \geq 1$ and $p+q = n-1$, and for the bilinear form $\mathcal{J}$ the linear form $\mathcal{J}(F,\cdot)$ vanishes identically.

Equivalently we consider the quotient tangent spaces $\widetilde{T}_xM^n = T_xM^n/span\{F(x)\}$ and the nondegenerate forms $\mathcal{J}$ on them. The derivative $D\Phi^t$ factors naturally to the quotient spaces and we can apply the theory developed previously to the derivative cocycles on the quotient spaces.

In particular if for a smooth flow the derivative cocycle is strictly $\mathcal{J}$–separated then by Theorem 4.3 we get a dominated splitting of the quotient space, but it does not necessarily lift to the splitting of the tangent space. We can lift the splitting in the case of strict $\mathcal{J}$–monotonicity (cf. [W1]) and we obtain the counterpart of the Lewowicz's Theorem 4.5.

Theorem 4.7. *If the derivative cocycle of a smooth flow is strictly $\mathcal{J}$–monotone then the flow is Anosov.*

5. Hyperbolic properties of W-flows.

We introduce a quadratic form $\mathcal{J}$ in the tangent spaces of the phase space of SM by $\mathcal{J}(\xi,\chi) = \langle \xi, \chi\rangle$. The form $\mathcal{J}$ factors naturally to the quotient bundle (the quotient by the span of the vector field (2.3), i.e., in the (ξ,χ) coordinates the quotient by the span of $(v,0)$), (note that it is so only for the restricted phase space SM). The quotient space can be represented by the subspace $\langle \xi, v\rangle = 0$, but this subspace is not invariant under the linearization of the flow (3.3) (or (3.4)). The form $\mathcal{J}$ in the quotient space is clearly nondegenerate and it has equal positive and negative indices of inertia.

We are going to calculate the Lie derivative of $\mathcal{J}$. The result is by necessity a form that factors onto the quotient bundle, like the $\mathcal{J}$ itself. Hence after (but not before) the differentiation we can consider ξ to be orthogonal to v. Using (2.1) and (3.3) we obtain

$$\frac{d}{dt}\mathcal{J}(\xi,\chi) = \frac{d}{dt}\langle \xi,\chi\rangle = \chi^2 \ - \varphi(v)\langle \xi,\chi\rangle - \langle \widehat{R}_a(\xi,v)v,\xi\rangle. \tag{5.1}$$

Theorem 5.1. *If the sectional curvatures of the Weyl structure are negative everywhere in M then the $W-$flow is strictly $\mathcal{J}$-separated and hence it has the dominated splitting.*

If for every plane Π the sectional curvatures satisfy

$$\widehat{K}(\Pi) + \frac{1}{4}E_\Pi^2 = K(\Pi) - div_\Pi E - E^2 + \frac{5}{4}E_\Pi^2 < 0,$$

where E_Π is the orthogonal projection of E on Π, then the $W-$flow is Anosov.

Proof. The first part of the claim follows immediately from (5.1) and Theorems 4.3. Indeed, by (5.1), for nonzero tangent vectors (ξ,χ) on the boundary of the positive cone $\mathcal{C}_+$ we have $\frac{d}{dt}\mathcal{J} > 0$. Which shows that the closure of the positive cone is taken under the flow into the positive cone, i.e., the flow is strictly $\mathcal{J}$-separated .

The second part of the claim follows from Theorem 4.7. Indeed we have

$$\frac{d}{dt}\mathcal{J}(\xi,\chi) = \left(\chi - \frac{1}{2}\langle E,v\rangle\xi\right)^2 + \left(-\frac{1}{4}\langle E,v\rangle^2 - \widehat{K}(\Pi)\right)\xi^2,$$

where the plane Π is spanned by v and ξ, (ξ is assumed to be orthogonal to v). We obtain that the W-flow is strictly $\mathcal{J}$-monotone if only $\widehat{K}(\Pi) + \frac{1}{4}E_\Pi^2 < 0$. Using the formula (3.6) we obtain the other expression.

□

In the case of negative sectional curvature let us denote the invariant subspaces of the dominant splitting in the quotient spaces by $\mathcal{E}^+(q,v)$ – the positive subspace and by $\mathcal{E}^-(q,v)$ – the negative subspace. We do not claim in general (nor do we expect) to have exponential growth in $\mathcal{E}^+(q,v)$ or exponential decay in $\mathcal{E}^-(q,v)$. We can represent these subspaces as graphs of operators defined on the tangent subspace orthogonal to v, i.e.,

$$\mathcal{E}^\pm(q,v) = \{(\xi,\chi) | \chi = U_\pm(q,v)\xi\}.$$

The reversibility of the flow ((2.4)) leads us to the conclusion that

$$U_\mp(q,v) = -U_\pm(q,-v). \tag{5.2.}$$

For 2-dimensional surfaces the negative curvature alone implies Anosov property.

Theorem 5.2. *For a 2-dimensional surface M if the curvature of the Weyl structure is negative, i.e, $\widehat{K} = K - divE < 0$ on M, then the W-flow is a transitive Anosov flow.*

Proof. Let us calculate the Lie derivative of χ^2. We get using (2.1) and (3.3)

$$\frac{d}{dt}\chi^2 = -2\langle \widehat{R}_a(v,\xi)v, \chi\rangle \tag{5.3}$$

In general we cannot claim that this derivative is positive in the positive cone of $\mathcal{J}$. In the case of a surface both χ and ξ can be assumed to be orthogonal to v and hence they must be colinear, i.e., $\chi = \frac{\langle\xi,\chi\rangle}{\xi^2}\xi$. Substituting this into (5.3) we obtain

$$\frac{d}{dt}\chi^2 = 2(-K + divE)\langle\xi,\chi\rangle, \tag{5.4}$$

Since our condition ensures that the W-flow is strictly $\mathcal{J}$-separated we have a continuous bundle of positive lines in the quotient spaces, invariant under the flow. Hence there is a positive continuous function u on SM such that the positive invariant line is described as $\chi = u\xi$. On the invariant line (5.4) yields

$$\frac{d}{dt}\chi^2 = \frac{2}{u}(-K + divE)\chi^2.$$

Since u is positive and bounded the last formula implies that there is exponential growth on the positive invariant lines.

By the reversibility (5.2) we conclude immediately that there is exponential decay on the negative invariant lines which proves that the system is Anosov.

The transitivity of the system follows from the observation made in Section 3 that we can always consider the field E to be divergence free and then the sectional curvature of the Weyl structure coincides with the Gaussian curvature. By assumption it is negative everywhere. Hence multiplying the vector field E by the parameter $a, 0 \leq a \leq 1$ we obtain a continuous family of Anosov flows ending with the geodesic flow on a surface of negative Gaussian curvature which is transitive. By the Anosov structural stability theorem we obtain that all these systems are topologically transitive. □

In higher dimensions under the assumption of negative sectional curvatures we obtain some hyperbolicity which is weaker than the Anosov property.

Theorem 5.3. *If the sectional curvatures of a Weyl structure are negative everywhere in M then there is uniform exponential growth of volume on the positive invariant subspace $\mathcal{E}^+$ and uniform exponential decay of volume on the negative invariant subspace $\mathcal{E}^-$.*

Proof. Since the W-flow is reversible ((2.4),(5.2)) it is enough to prove the exponential growth on $\mathcal{E}^+$. The crucial element in the proof is the choice of the "Lyapunov volume element" on the invariant subspaces which is increased under the dynamics.

For convenience we employ the coordinate form (3.4) of the linearized equations. We can represent $\mathcal{E}^+$ as the graph of an operator $U : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ with positive definite symmetric part,

$$\mathcal{E}^+ = \{(\xi, \chi) | \widetilde{\chi} = U\widetilde{\xi}\}.$$

The operator U satisfies the Riccati equation

$$\frac{d}{dt}U = \varphi(v)U - U^2 - R, \quad \text{where} \quad R\widetilde{\xi} = \widehat{R}_a(\xi, v)v. \tag{5.5}$$

Note that in contrast to the Riemannian case the operator R is not symmetric and hence the operator U cannot be expected to be symmetric. We split the operators $U = U_s + U_a$ and $R = R_s + R_a$ into the symmetric and antisymmetric parts. For the symmetric part U_s we obtain from (5.5)

$$\frac{d}{dt}U_s = \varphi(v)U_s - U_s^2 - U_a^2 - R_s. \tag{5.6}$$

By the assumption of negative sectional curvatures the operator $-R_s$ is positive definite. Moreover since U_a is antisymmetric $-U_a^2$ is positive semidefinite.

We introduce new linear coordinates $\kappa \in \mathbb{R}^{n-1}$ on $\mathcal{E}^+$ by the formula $\kappa = U_s\widetilde{\xi}$. Using (3.4) and (5.6) we obtain

$$\frac{d}{dt}\kappa = \left(-U_s + U_sUU_s^{-1} + (-U_a^2 - R_s)U_s^{-1}\right)\kappa.$$

Since $tr(U_sUU_s^{-1}) = trU_s$ the trace of the operator in right hand side of this linear equation is equal to $tr[(-U_a^2 - R_s)U_s^{-1}] > 0$ and we obtain that the standard volume element in the coordinates κ is uniformly exponentially expanded.

□

Corollary 5.4. *If the sectional curvatures of the Weyl structure are negative everywhere in M then for any ergodic invariant measure of the W-flow the largest Lyapunov exponent is positive and the smallest Lyapunov exponent is negative.*

We can apply this corollary to an individual periodic orbit and we obtain linear instability. Moreover there are also no linearly repelling or linearly neutral periodic orbits.

In the case of a potential field $E = -\nabla U$ the Weyl connection is the Levi-Civita connection of the Riemannian metric $\widetilde{g} = e^{-2U} g$. For such flows we can rewrite (5.1) as

$$\frac{d}{dt}\left(e^{-U}\mathcal{J}(\xi,\chi)\right) = e^{-U}\left(\chi^2 \; - \langle \widehat{R}_a(\xi,v)v,\xi\rangle\right).$$

This allows us to make a stronger claim for some periodic orbits than Corollary 5.4.

Proposition 5.5. *If the potential U is single valued on a periodic orbit of a W-flow and the sectional curvatures are negative then the periodic orbit is hyperbolic and it has half of the Floquet multipliers outside of the unit circle and half of them inside.*

In the 2-dimensional case we have $divE = -\triangle U$ and Theorem 5.1 leads directly to

Corollary 5.6. *If $K < -\triangle U$ on a 2-dimensional surface M then the W – flow is a transitive Anosov flow.*

Corollary 5.7. *If the vector field E has a local potential $E = -\nabla U$ which is harmonic and the Gaussian curvature $K < 0$ on M then the W – flow is a transitive Anosov flow.*

We can conclude that in the case of fields given by automorphic forms on surfaces of constant negative curvature, which were studied by Bonetto, Gentile and Mastropietro, [B-G-M], the flow is always Anosov. It was shown in [W1] that once such a flow is Anosov it is also automatically dissipative, i.e., the SRB measure is singular. Note that in this situation we can multiply the vector field E by an arbitrary scalar and we still get a transitive Anosov flow. It would be interesting to understand the asymptotics of the SRB measure as $\lambda \to \infty$. Is the limit supported on the union of integral curves of E ? Let us stress that this scenario differs from the perturbative conditions in [Go], [Gr], [W1], where the geodesic curvature of the trajectories cannot be too large. Our trajectories may have arbitrarily large geodesic curvatures and yet give a transitive Anosov flow.

The following conjecture is of considerable interest

Conjecture 5.8. *There are manifolds of dimension ≥ 3 and 1-forms φ such that the Weyl sectional curvatures are negative everywhere but the W-flow is not Anosov.*

It is also plausible that under the assumption of negative sectional curvatures we can obtain W-flows which are nontransitive Anosov flows, as in the examples of Franks and Williams, [F-W].

To construct such examples we need to find nice examples of Weyl manifolds with negative sectional curvatures. In that direction there are some notable obstructions.

Proposition 5.9. *There are no Weyl structures with negative sectional curvatures in a small neighborhood of the structure defined by the constant vector field E on a flat torus (Example 2.2) or in a small neighborhood of the structure defined in Example 2.3.*

Proof. As noted in the discussion of Examples 2.2, 2.3, the W-flow has two invariant submanifolds. Let us consider the attracting submanifold. It is defined by the equation $|E|v = E$. Its tangent bundle is given by $\chi = |E|(\xi - \langle \xi, v\rangle v)$. On the invariant manifold the linearized equations (3.4) read

$$\frac{d\xi_0}{dt} = 0, \quad \frac{d\widetilde{\xi}}{dt} = -|E|\widetilde{\xi} + \widetilde{\chi}, \quad \frac{d\widetilde{\chi}}{dt} = 0.$$

The bundle defined by $\chi = 0$ is an invariant normal bundle. On this bundle we have exponential decay while on the invariant submanifold the dynamics is by isometries. Hence the W-flow is normally contracting at the invariant submanifold. It follows that the invariant manifold persists under small perturbations (cf. [Ro], p.445). The invariant submanifold of the perturbed system is C^1 close to the original one.

The tangent subspaces of the unperturbed invariant submanifold are in the positive cone of the form $\mathcal{J}$ and so are the tangent subspaces of the perturbed invariant submanifold, at least when the perturbation is sufficiently small. Hence for such perturbations, if the sectional curvatures are negative then the invariant submanifold must be tangent to $\mathcal{E}^+ \oplus span\{(v,0)\}$ (cf. (4.1)). But on this bundle we have by Theorem 5.4 uniform exponential growth of the volume element. It is incompatible with the compactness of the invariant manifold. □

We do not know, if the d-dimensional, $d \geq 3$, torus carries a Weyl structure with negative sectional curvatures.

Example 5.10.

Let us consider a vector field E on the two dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$ with finite number of points $\{s_1, s_2, \dots, s_N\}$ removed. We allow the field to have singularities at these points. We assume that E has a local potential U which is superharmonic on the punctured torus, $\triangle U < 0$. We can represent $U = U_0 + U_1$ where U_0 is a linear function and U_1 is a doubly periodic function with singularities which is superharmonic away from the singularities. Such a function with only one singularity can be produced by taking a superharmonic function in $\mathbb{R}^2$ with one singularity and with fast decay at infinity, e.g., the function f given in polar coordinates (r, α) by $f = \ln r - \ln(e^{-r} + r)$. Then we take the sum over the integer translates

$$U_1(q) = \sum_{a \in \mathbb{Z}^2} f(q + a).$$

This function has the desired properties.

Let us consider the W-flow with such a potential U. Locally, up to the time change, it is the geodesic flow of the metric $e^{-2U}ds^2$ where ds^2 denotes the flat metric of the torus. Such a flow is in general incomplete, moreover open sets of orbits may end up at the singularities. This can be excluded by allowing only logarithmic singularities.

Proposition 5.11. *With the above assumptions on the local potential* U*, if at every singularity point* s *we have* $U_1(q) = \beta \ln r + B(q)$ *for* $0 < \beta < 1$ *and some smooth function* $B(q)$ *defined in a neighborhood of* s*, with* $r = |q - s|$*, then for every point on the punctured torus the trajectories of the W-flow are defined globally except for countably many directions which result in the trajectories reaching one of the singularities in finite time.*

Proof. Let us assume that $s = 0$ and $q \in \mathbb{C}$. We lift the metric $e^{-2U}ds^2$ to the Riemann surface of the function $w = q^\gamma$, $\gamma > 0$. We obtain the metric

$$\frac{1}{\gamma}|w|^{\frac{2(1-\gamma-\beta)}{\gamma}} e^{-2U_0(w^{\frac{1}{\gamma}})-2B(w^{\frac{1}{\gamma}})} dw d\bar{w}.$$

Choosing $\gamma = 1 - \beta$ we obtain a finite metric on the Riemann surface with the boundary $w = 0$. We do not have differentiability at the boundary but we know that inside the curvature is nonpositive. This allows us to conclude that through every point on the surface there is at most one geodesic that reaches the boundary $w = 0$. $\square$

For such special singularities we obtain a measurable flow with hyperbolic properties. It seems that the results of Chernov [Ch] could be extended to cover these flows and then we would obtain a unique SRB measure. It is definitely so for special values of $\beta_i = 1 - \frac{1}{k_i}$ with k_i a positive integer. For these values of β-s we obtain a smooth system on a branched covering of the torus. This construction was used before by Knauf, [Kn], in his study of motion in the fields with Coulombic singularities.

Theorem 5.12. *For the W-flow* $\Psi^t : S\mathbb{T}^2 \to S\mathbb{T}^2$ *described in Proposition 5.11, if* $\beta_i = 1 - \frac{1}{k_i}$ *with* k_i *equal to positive integers,* $i = 1, \dots, N$*, then there exists a unique SRB measure* ν_0 *on* $S\mathbb{T}^2$*, i.e., an invariant measure such that*

a. for every measure ν *with positive density with respect to the Lebesgue measure*

$$\Psi^t_*(\nu) \to \nu_0$$

weakly as $t \to +\infty$*.*

b. for every continuous function f *on* $S\mathbb{T}^2$ *we have for Lebesgue almost all* $x \in S\mathbb{T}^2$

$$\frac{1}{T}\int_0^T f(\Psi^t x)dt \to \int_{S\mathbb{T}^2} f(x)d\nu_0(x).$$

Moreover, if the linear part U_0 *of the potential* U *is nonzero then the measure* ν_0 *is singular with respect to the Lebesgue measure.*

Proof. We split the potential U into the linear and the periodic parts, $U = U_0 + U_1$. We remove the periodic part by changing the metric to $e^{-2U_1}ds^2$. By the construction of Knauf [Kn] we can lift this metric to a smooth metric of negative Gaussian curvature on a branched covering of $\mathbb{T}^2$. On this branched covering we get the smooth W-flow with the local potential equal to the lift of U_0. The lift of U_0 is a harmonic function on the surface and hence by Corollary 5.8 the W-flow is a transitive Anosov flow. Now the theorem follows from the general theory of SRB measures for transitive Anosov flows, [S][B-R]. (Note that the conclusions of our theorem are not effected by the time change in passing from one metric to the other.) It was shown in [W1] that the SRB measure has to be singular, if the potential is not global.

□

Another way to obtain hyperbolic W-flow is to enclose the singularities of a superharmonic potential on a torus in round scatterers and assume elastic reflections from the scatterers. We will discuss these models in the next section.

§6. Billiard W – flows.

Let us now assume that the Riemannian manifold M has a boundary. Given a vector field E we define the W – flow with collisions by the condition that when a trajectory of our flow reaches the boundary the velocity is reflected by the rules of elastic collisions. We can for instance consider a closed manifold without a boundary and remove certain number of open sets with smooth boundaries ("obstacles" or "scatterers"). It is clear that we obtain in this way a flow with collisions, as defined in [W6]. Such systems have discontinuities which are associated with tangencies of the trajectories of the W-flow to the boundary and any singularities (corners) of the boundary. The orbits that do not experience any discontinuity will be called smooth.

We will restrict our attention to billiard W-flows such that on all smooth orbits there are infinitely many collisions with the boundary and the time between collisions is uniformly bounded away from zero and infinity (finite horizon). In particular we assume that the boundaries have no corners. For such flows we can introduce also the Poincare section of the billiard W-flow, *billiard map* for short, defined on the manifold of "outward" unit tangent vectors over the boundary.

The description of the linearized dynamics is similar to the case of Riemannian billiards. Let us consider a scatterer O and a point $q_0 \in O$. We denote by N the field of unit vectors orthogonal to the boundary O and pointing inwards (of the manifold M). The collision map is described by the formula

$$v^+ = v^- - 2\langle v^-, N\rangle N = Zv^-, \tag{6.1}$$

where v^+ denotes the velocity immediately after, v^- the velocity immediately before the collision and Z denotes the reflection in the tangent subspace to O. To obtain the linearized dynamics let us consider a family of trajectories of the W-flow $q(t,u), v(t,u) = \frac{dq}{dt}, |u| < \epsilon$ coming to a collision with O in the vicinity of $q_0 = q(t_0, 0)$. At the moment of collision $t_0 = t_0(u)$ the velocity is changed by the reflection in the tangent plane of O described by (6.1).

Let us denote by $P_\pm$ the projection onto the tangent subspace of O along the velocity vector $v^\pm$. We introduce $\xi_0 = \frac{d}{du}q(t_0(u),u)_{|u=0} = P_-\xi^- = P_+\xi^+$, where $\xi^\pm$ are the values of the Jacobi field $\xi = \frac{dq}{du}$ immediately before and immediately after the collision. We have clearly $\xi^+ = Z\xi^-$. To calculate the change in the χ coordinate let us recall (Proposition 3.3) that $\chi^\pm$ is the component of $\widehat{\nabla}_{\xi^\pm} v$ orthogonal to v at time t_0. Hence we can replace $\xi^\pm$ in the derivative by ξ_0, i.e., $\chi^\pm = \widehat{\nabla}_{\xi_0} v^\pm - \langle \widehat{\nabla}_{\xi_0} v^\pm, v^\pm\rangle v^\pm$.

Proposition 6.1.

$$\chi^+ = Z\chi^- - 2\langle v^-, N\rangle P_+^* \widehat{K}\xi_0, \tag{6.2}$$

where $\widehat{K}\xi_0 = \widehat{\nabla}_{\xi_0} N - \langle \xi_0, E\rangle N = \nabla_{\xi_0} N + \langle N, E\rangle \xi_0$.

Note that the operator $\widehat{K}$ is the orthogonal projection of the " Weyl shape operator" $\widehat{\nabla}_{\xi_0} N$ to the boundary O.

Proof. Differentiating (6.1) we obtain

$$\begin{aligned}\widehat{\nabla}_{\xi_0} v^+ =& \widehat{\nabla}_{\xi_0} v^- - 2\langle \widehat{\nabla}_{\xi_0} v^-, N\rangle N - \\ & - 2\langle v^-, \widehat{\nabla}_{\xi_0} N\rangle N - 2\langle v^-, N\rangle \widehat{\nabla}_{\xi_0} N + 4\langle \xi_0, E\rangle\langle v^-, N\rangle N = \\ =& Z\widehat{\nabla}_{\xi_0} v^- - 2\langle v^-, \widehat{K}\xi_0\rangle N - 2\langle v^-, N\rangle \widehat{K}\xi_0,\end{aligned}$$

□

If we denote by $K\xi_0 = \nabla_{\xi_0} N$ the Riemannian shape operator of the boundary O we have

$$\widehat{K}\xi_0 = K\xi_0 + \langle N, E\rangle \xi_0. \tag{6.3}$$

We say that the submanifold O is (strictly) Weyl concave if $\widehat{K}$ is positive (definite) semidefinite.

We want to establish what happens at collisions to the quadratic form $\mathcal{J}(\xi, \chi) = \langle \xi, \chi\rangle$, introduced in the previous section.

Proposition 6.2. *The billiard W-flow on a manifold M of nonpositive sectional Weyl curvature and (strictly) Weyl concave boundary O is (strictly) $\mathcal{J}$-separated.*

Proof. At the collision we have by Proposition 6.1

$$\mathcal{J}(\xi^+, \chi^+) = \mathcal{J}(\xi^-, \chi^-) - 2\langle v^-, N\rangle\langle \widehat{K}\xi_0, \xi_0\rangle. \tag{6.4}$$

□

The fact that the flow is strictly $\mathcal{J}$-separated does not allow to construct the dominated splitting on every smooth orbit, as in Theorem 4.3. The problem is that the coefficients of contraction on $\mathcal{L}_+$ from Theorem 4.2 are not bounded away from 1, except in dimension 2. We saw in Section 5 the special status of the 2-dimensional case. Similarly to Theorem 5.1 we obtain.

Theorem 6.3. *If the Weyl curvature of a 2-dimensional surface M with the boundary O is nonpositive and the boundary O is strictly Weyl concave then the billiard map has the dominated splitting on every smooth orbit and there is uniform expansion on the positive invariant line and uniform contraction on the negative invariant line.*

Proof. To construct the dominated splitting we need to estimate the coefficients of contraction on $\mathcal{L}_+$.

The derivative of the billiard map is the composition of the derivative (6.2) of the collision map and the derivative of the W-flow between collisions. We will be using the orthonormal frames $e_0(t) = v(t), e_1(t)$, which were used in (3.4). We continue these frames across collisions by the reflection Z in the boundary.

In the 2-dimensional case the relation (6.2) is scalar. Let us denote by k and $\widehat{k} = k + \langle E, N\rangle$ the Riemannian geodesic curvature and the "Weyl geodesic curvature" of O respectively. By the assumption of strict concavity $\widehat{k}$ is bounded away from zero.

When ξ^- is in the plane spanned by v^- and N then

$$P_+^*\xi_0 = \frac{1}{\langle v^-, N\rangle^2} Z\xi^-.$$

Hence we obtain $\chi^+ = Z(\chi^- + h\xi^-)$ where $h = -2\frac{\widehat{k}}{\langle v^-, N\rangle} > 0$. Taking into account that $\xi^+ = Z\xi^-$ we conclude that the derivative of the collision map is described by the matrix $\begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix}$, with the coefficient h bounded away from zero, and unbounded.

Between collisions we get from (3.4)

$$\frac{d\xi}{dt} = -g(t)\xi + \chi, \quad \frac{d\chi}{dt} = -r(t)\xi, \tag{6.5}$$

where ξ, χ are scalars, and $g(t) = \langle E(t), v(t)\rangle, r(t) = \langle \widehat{R}_a(\xi, v)v, e_1\rangle$. The contribution into the derivative of the billiard map is described by the fundamental matrix solution of these equations. Let us denote this contribution by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since the dynamics is $\mathcal{J}$-separated the matrix has nonnegative entries. By compactness the determinant of this matrix is bounded away from zero and from infinity, and all the entries are bounded. It also follows from the form of the equations (6.5) that b is bounded away from zero.

The derivative of the billiard map is given by the product of these matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix} = \begin{pmatrix} a+hb & b \\ c+hd & d \end{pmatrix}.$$

The coefficient of contraction on $\mathcal{L}_+$ appearing in Theorem 4.2 is equal to

$$\frac{r_1^-}{r_1^+} = \frac{\sqrt{(a+hb)d} - \sqrt{(c+hd)b}}{\sqrt{(a+hb)d} + \sqrt{(c+hd)b}} = \frac{\sqrt{A}-1}{\sqrt{A}+1},$$

where $A = \frac{(a+hb)d}{(c+hd)b} \le \frac{a}{hb} + 1$. Since a is bounded and h, b are bounded away from zero we conclude that the coefficient of contraction is bounded away from 1. Consequently on every smooth orbit we obtain the two invariant line bundles. Let us describe the positive line bundle by $\chi = u\xi$. Clearly

$$\frac{c+hd}{a+hb} < u < \frac{d}{b},$$

so that u is bounded away from zero and infinity. (Let us note that u is bounded because in our billiard map the reflection in the boundary goes first. It is not so if we consider the billiard map in which we start with the flow.)

As in the proof of Theorems 5.1 and 5.2 we calculate now the change in χ^2 at the collision. On the positive invariant line $\chi^- = u\xi^-$ we obtain

$$(\chi^+)^2 = (\chi^-)^2(1+\frac{h}{u})^2.$$

Since u was established to be bounded and χ^2 does not decrease between collisions we obtain the desired uniform expansion.

□

In dimension 3 and higher we do not obtain the dominated splitting on all orbits since we do not get the required uniform estimates at the collision with the boundary. We do get the dominated splitting on individual periodic orbits or more generally for almost all orbits of some ergodic invariant measure as guaranteed by Theorem 4.6. Moreover parallelly to Theorem 5.3 we obtain exponential growth and decay of volumes on the invariant subspaces. This is the content of the next theorem. We will consider only the ergodic invariant measures for which the derivative of the billiard map satisfies the integrability condition which guarantees the existence of Lyapunov exponents. In special examples it may be difficult to establish such integrability for singular invariant measures, since the collisions lead to unbounded derivatives.

Theorem 6.4. *If the Weyl sectional curvatures of a manifold M are nonpositive everywhere and the boundary S is strictly Weyl concave, then for any ergodic invariant measure of the billiard flow the largest Lyapunov exponent is positive and the smallest Lyapunov exponent is negative.*

Proof. We follow the scheme of the proof of Theorem 5.3. We obtain the invariant bundles of the dominated splitting defined almost everywhere. We need to study the growth at collisions of the volume element on the positive invariant subspaces introduced in the proof of Theorem 5.3. On the positive subspace $\mathcal{E}^+$ given by $\chi^\pm = U^\pm\xi^\pm$ we can rewrite (6.2) as

$$\chi^+ = Z\chi^- + H\xi^+, \quad \text{where} \quad H = -2\langle v^-, N\rangle P_+^*\widehat{K}P_+.$$

The operator H is symmetric and positive definite. It follows from the above formula that $U^+ = ZU^-Z^{-1} + H$. As in Theorem 5.3 the operators U are not in general symmetric but their symmetric parts U_s are positive definite and $U_s^+ = ZU_s^-Z^{-1} + H$. For $\kappa = U_s\xi$ we get

$$\kappa^+ = (ZU_s^-Z^{-1} + H)Z(U_s^-)^{-1}\kappa^-.$$

We observe that

$$|\det(ZU_s^-Z^{-1} + H)Z(U_s^-)^{-1}| = \frac{\det(U_s^- + Z^{-1}HZ)}{\det U_s^-} > 1.$$

We cannot guarantee that the determinant is bounded away from 1 but using the ergodic theorem we obtain the exponential growth of volume on almost all orbits. Note that we used the fact established in the proof of Theorem 5.3 that between collisions the standard volume element in κ coordinates is not decreased. □

As in the case of W-flows we can apply this theorem to individual periodic orbits and we obtain that all such orbits are linearly unstable and there are no repelling orbits. In the case of the Weyl structure with a local potential we can formulate a counterpart of Proposition 5.5 because by (6.4) the form $\mathcal{J}$ is not decreased at the collisions.

Proposition 6.5. *For a locally potential Weyl structure with nonpositive sectional curvatures and strictly Weyl concave boundary, if the potential is single valued on a periodic orbit of the billiard W-flow then the periodic orbit is hyperbolic and it has half of the Floquet multipliers outside of the unit circle and half of them inside.*

Let us consider the case of a a 2-dimensional flat torus and the harmonic local potential U, i.e., $\triangle U = 0$. We have calculated in Section 3 that in such a case the Weyl curvature is zero. It turns out that we can also integrate the equations of the W-flow. We take the torus $\mathbb{T}^2 = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$. Let $f(z), z = x + iy$, be the holomorphic function such that $U = Re(f)$

Proposition 6.5. *The conformal mapping defined by*

$$F(z) = \int e^{-f(z)} dz$$

takes the trajectories of the W-flow into straight lines. In particular, if $z(t)$ is a trajectory (parametrized by the arc length of the flat background metric) then $e^{-iIm(f)}z' = const.$

Proof.

$$dFd\bar{F} = e^{-f}dze^{-\bar{f}}d\bar{z} = e^{-2U}(dx^2 + dy^2).$$

We know that the trajectories of the W-flow are geodesics of the last metric which proves the first claim.

To prove the second claim we observe that $F(z(t))$ is a straight line and hence $\frac{d}{dt}F(z(t))$ has a constant direction.

□

Now we can easily analyze the linear stability of periodic orbits of the W-flow by mapping them onto periodic orbits of a planar billiard. For planar billiards the linear stability of an orbit is determined by the lengths of the segments of the orbit, the angles at which they reflect from the boundaries and the curvatures of the boundaries. The angles do not change under the conformal map and the length of the image of a trajectory from $z_0 = z(t_0)$ to $z_1 = z(t_1)$ is equal to $|F(z_1) - F(z_0)|$. It remains to express the geodesic curvature of the image of the boundary. Let $z(s)$ be a parametrized curve oriented by the field of normals $N = i\frac{z'}{|z'|}$. The signed curvature of the curve is equal to $k = -\frac{1}{|z'|}Im(\frac{z''}{z'})$. By straightforward calculation we obtain that the signed curvature $\widetilde{k}$ of the image curve $F(z(s))$ is equal to

$$\widetilde{k} = \frac{1}{|F'|}\left(k + Im\left(\frac{F''z'}{F'|z'|}\right)\right). \tag{6.6}$$

Finally let us consider the system with round scatterers and constant external field, studied by Moran and Hoover , [M-H], and Chernov, Eyink, Lebowitz and Sinai, [Ch-E-L-S]. We put $U = -ax$ and then we get symmetric periodic orbits shown in Fig.1. It turns out that the "flat" orbit is never linearly stable and the "curved" orbit is linearly stable as soon as the Weyl curvature of the scatterers at the points of reflections becomes positive. Hoover and Moran observed numerically these stable orbits and the KAM "bottles" enclosing them.

The stability of an orbit depends on its length l and the signed curvatures at the points of reflections k_0, k_1 (k is negative for a strictly concave boundary). The orbit is linearly stable if and only if (cf. [W7])

$$lk_0k_1 - k_0 - k_1 < 0, \quad \text{and} \quad (lk_0 - 1)(lk_1 - 1) > 0. \tag{6.7}$$

We have $F(z) = \frac{1}{a}e^{az}$. Let us assume that the points of reflection are $z_0 = x_0 + iy_0$ and $z_1 = x_1 + iy_1$ Using Proposition 6.5 and (6.6) we obtain for the flat orbit in Fig 1 $(y_0 = y_1, x_0 < x_1)$

$$l = \frac{1}{a}(e^{ax_1} - e^{ax_0}), \quad k_0 = e^{-ax_0}(-\frac{1}{r_0} - a), \quad k_1 = e^{-ax_1}(-\frac{1}{r_1} + a),$$

where r_0 and r_1 are the radii of the scatterers (which we allow to be different). It follows that

$$lk_0k_1 - k_0 - k_1 = \frac{1}{ar_0r_1}(e^{-ax_0} - e^{-ax_1}) + \frac{1}{r_0}e^{-ax_1} + \frac{1}{r_1}e^{-ax_0} > 0.$$

Hence the orbit is always unstable, even if the scatterers have different radii.

On the contrary for the curved symmetric orbit, $ax = -\ln\cos ay, \quad y_1 = -y_0 > 0, r_0 = r_1 = r, k_0 = k_1 = k$, in Fig.1 we have

$$l = \frac{1}{a}|\tan ay_1 - \tan ay_0|, \quad k = \cos ay_1(-\frac{1}{r} + a\sin ay_1).$$

The second condition in (6.7) is satisfied automatically, while the first reads $k(lk-2) < 0$. It is satisfied if and only if the curvature k is positive (i.e., $ar\sin ay_1 > 1$). Indeed we have $lk = 2\sin ay_1(-\frac{1}{ar} + \sin ay_1) < 2$,

In dimension 3 and higher the sectional curvatures are nonpositive and we still get the types of periodic orbits considered above. Since the potential on these periodic orbits is singlevalued we can apply Proposition 6.5 and we obtain hyperbolicity of these orbits under the assumption that at the collision points the boundary is Weyl strictly concave.

By (6.3) the boundary is Weyl strictly concave if $r|E| < 1$ where r is the radius of a scatterer. Under this assumption and the assumption of finite horizon we get complete hyperbolicity in dimension 2 by Theorem 6.3. It sets the stage for the study of existence and properties of SRB measures under such assumption. It seems that the results of Chernov [Ch] could be generalized to include this case. Further we expect that in [Ch-E-L-S] the assumption of weak field can be replaced by $r|E| < 1$, at least for some parts of the theory.

In dimension 3 and higher with this assumption we obtain by Theorem 6.4 at least one positive Lyapunov exponent and one negative Lyapunov exponent for "nice" invariant measures. The issue of complete hyperbolicity requires further study.

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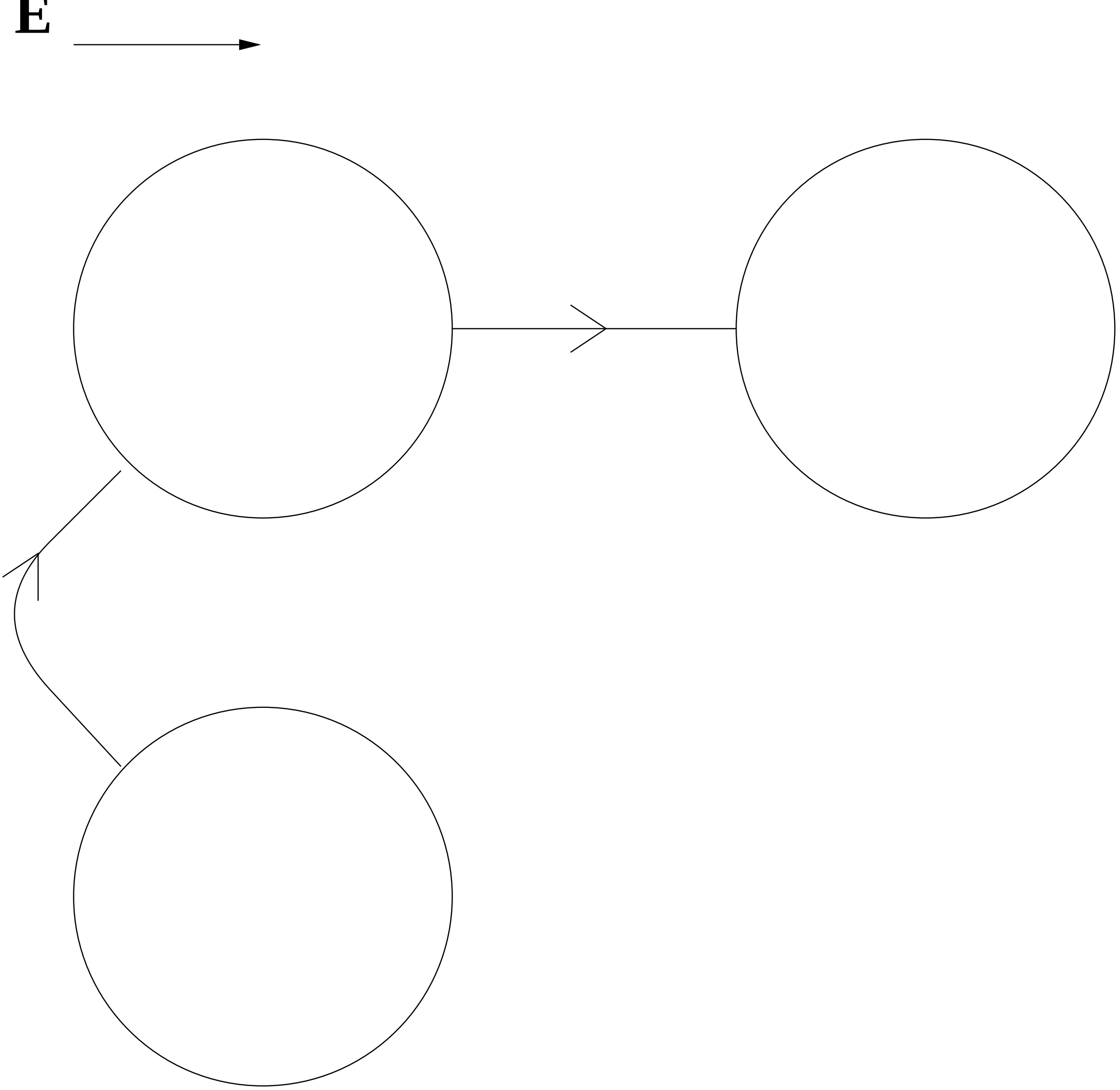

Fig 1. Periodic orbits in the Lorentz gas with an external field E and the Gaussian thermostat, "flat" and "curved".