COMPLETE HYPERBOLICITY IN HAMILTONIAN SYSTEMS WITH LINEAR POTENTIAL AND ELASTIC COLLISIONS

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ABSTRACT. We present a class of systems with all Lyapunov exponents different from zero (completely hyperbolic). They are obtained by the restriction of the configuration space of a simple completely integrable system with linear potential and elastic collisions. We show that special geometry of the configuration space is necessary. We survey concrete realizations of this scheme discussed previously in [W6] and elsewhere. **Published in Reports on Mathematical Physics, Vol 44 (301 – 312), 1999.**

§1.Introduction.

There are few systems known to be Anosov (more generally hyperbolic, possibly nonuniformly and partially, in all of the phase space), among physically natural dynamical systems. Recent work of Gallavotti [G] brings out the importance of this type of behavior for nonequilibrium statistical mechanics.

There are numerous examples of systems where hyperbolicity occurs in some part of the phase space, i.e. there is an invariant subset with hyperbolicity, for instance the Smale horseshoe associated with splitting of separatrices. Such type of phenomena will not be discussed in this paper.

To establish hyperbolic behavior in all of the phase space we need to have a system in which it is indeed present in all of the phase space. The method we are going to discuss cannot tell us that a system is *not* hyperbolic, it gives either a positive or inconclusive answer.

Hyperbolicity is defined in the language of the linearized system (the derivative in case of discrete time). It requires establishing exponential growth or decay of solutions of the linearized system. Since the solutions will grow or decay depending on the initial conditions it is apparent that we need to control the directions of the solutions and not their magnitude alone. The main method is to use indefinite quadratic forms. If an indefinite quadratic form (more precisely a field of such forms on the tangent bundle of the phase space) evaluated on any nonzero solution grows then it provides us with a distinguished cone (sector) of positive tangent vectors which is taken into itself by the linearized dynamics.

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Once a solution enters this cone then the quadratic form can be used as a measure of the magnitude of the solution and the growth can be usually strengthened to exponential growth with the help of recurrence. The decay of some solutions is obtained by reversing time and using the negative of our quadratic form.

This crucial idea was put forward by Lewowicz [L] and independently in [W1].

In the Hamiltonian case the scheme has some special features which were formulated in [W5], see also [W2] and [L-W]. In §2 we will briefly describe the symplectic criterion for hyperbolicity of Hamiltonian systems. It involves monotonicity of the linearized dynamics with respect to a quadratic form defined by a pair of transversal Lagrangian subspaces.

The strict monotonicity was found so far in few systems. In §3 we describe a simple completely integrable Hamiltonian system with linear potential and elastic collisions which can be transformed into strictly monotone system by the restriction of the configuration space. The occurrence of strict monotonicity is limited by algebraic conditions which lead to some interesting geometry of simplicial cones (wedges). It maay look surprising that hyperbolicity is guaranteed by rigid algebraic conditions. One would expect to have open conditions in some functional spaces. Possible explanation is that with nonuniformly hyperbolic systems even small perturbations may produce elliptic periodic orbits. Hence only special deformations are allowed that guarantee elliptic free behavior. As a result in this situation the search for monotone systems resembles the search for complete integrability, which is easily destroyed by a small perturbation. Naturally small perturbations may still be completely hyperbolic on most of the phase space. By KAM theory small perturbations of completely integrable systems are completely integrable except on a subset of small Lebesgue measure. Such a counterpart of KAM theory for nonuniform hyperbolic behavior is missing.

In the rest of the paper we discuss concrete applications of the scheme of §3.

In §4 we consider the dynamics in 2-dimensional wedges, We demonstrate an interesting system in which there is a coexistence of hyperbolic components with a packet of invariant tori. The packet becomes thinner and thinner as the energy increases, until at the critical level only one invariant torus remains, dividing the energy shell into two components. For even bigger energies the two components coalesce.

In §5 we survey recent results on many particle systems in a line interacting by constant forces obtained in [W6].

§2 The Q-criterion for hyperbolicity.

We will outline here a criterion for nonvanishing of Lyapunov exponents. Complete exposition can be found in [W5].

The $2n$-dimensional phase space M of a Hamiltonian system is endowed with the symplectic structure, a nondegenerate closed 2-form ω. Let us denote by F a Hamiltonian vector field on M, and by $\Psi^t, t \in \mathbb{R}$, the Hamiltonian flow in M, defined by a hamiltonian function H.

The n-dimensional tangent subspaces on which ω vanishes sre called Lagrangian. We choose two transversal subbundles, $L_1(x)$ and $L_2(x), x \in M$, of Lagrangian subspaces in the tangent bundle of M. We allow these bundles to be discontinuous and defined almost everywhere. The only requirement is their measurability.

An ordered pair of transversal Lagrangian subspaces, L_1 and L_2, defines a quadratic form $\mathcal{Q}$ by the formula

$$\mathcal{Q}(w) = \omega(w_1, w_2), \quad \text{where } w = w_1 + w_2, w_i \in L_i, i = 1, 2.$$

Further we define the symplectic sector $\mathcal{C}$ between L_1 and L_2 by $\mathcal{C} = \{w | Q(w) \geq 0\}$.

We assume that F belongs to L_2 at almost all points (we could as well assume that it belongs to L_1). This assumption is very important for the Hamiltonian formalism. It leads to $\mathcal{Q}(w + \lambda F) = \mathcal{Q}(w)$, for all scalars λ and all vectors w tangent to the level set of the hamiltonian, $dH(w) = 0$. This property allows to project the quadratic form $\mathcal{Q}$ on the factor of the tangent space to the level set of the hamiltonian by the one dimensional subspace spanned by F. This factor space plays the role of the "transversal section" of the linearized flow restricted to a smooth level set of the hamiltonian. Note that in general we do not have an invariant codimension one subspace transversal to the flow.

Definition 2.1. *The Hamiltonian flow, Ψ^t, is called monotone (with respect to the bundle of symplectic sectors $\mathcal{C}(x), x \in M$) or $\mathcal{Q}$-monotone, if for almost all points in M*

$$\mathcal{Q}(D\Psi^t w) \geq \mathcal{Q}(w),$$

for all $t \geq 0$ and all vectors w tangent to a level set of the hamiltonian ($dH(w) = 0$).

Definition 2.2. *A Hamiltonian flow with collisions is called (nonuniformly) partially hyperbolic, if it has nonzero Lyapunov exponents almost everywhere, and it is called (nonuniformly) completely hyperbolic, if it has all but two nonzero Lyapunov exponents almost everywhere.*

Completely integrable Hamiltonian systems have only zero Lyapunov exponents.

The monotonicity of a Hamiltonian flow does not imply nonvanishing of any Lyapunov exponents. Actually completely integrable Hamiltonian flows are typically monotone with respect to some bundle of symplectic sectors. To obtain hyperbolicity one needs to examine what happens to the "sides" L_1 and L_2 of the sector $\mathcal{C}$, or equivalently to what extent the inequality in Definition 2.1 is strict.

Roughly speaking, if L_1 enters the symplectic sector completely after some time and L_2 enters it except for a k dimensional subspace then we have $(n - k)$ positive Lyapunov exponents. That is the essence of the $\mathcal{Q}$-criterion for hyperbolicity. Precise formulation and the proof can be found in [W5]

§3 Hamiltonian system with linear potential.

We consider the system with the hamiltonian $H = \sum_{i=1}^{n} \left(\frac{v_i^2}{2} + a_i x_i \right)$, where x_i are position variables, v_i are velocities and the constants $a_i > 0$. We restrict the configuration space to the positive orthant $\mathcal{O}^+ = \{x \in \mathbb{R}^n | x_1 \geq 0, \ldots, x_n \geq 0\}$ by introducing elastic collisions with the coordinate hyperplanes $\{x_i = 0\}$. Such a system is closed (every energy shell is compact) and it is completely integrable (the quantities $h_i = \frac{v_i^2}{2} + a_i x_i$ are conserved both in the continuous time dynamics and in collisions).

This simple system is monotone with the following choice of the two Lagrangian subspaces

$$L_1 = \{dv_1 = dv_2 = \cdots = dv_n\}, \quad L_2 = \{dh_1 = dh_2 = \cdots = dh_n\}.$$

The symplectic sector between them is the positive cone of the quadratic form $\mathcal{Q} = \sum_i dx_i dv_i + \sum_i \frac{v_i}{a_i} dv_i^2$. Indeed the equations of motion and their linearization are

$$\begin{aligned} \frac{dx_i}{dt} &= v_i, & \frac{d}{dt}dx_i &= dv_i, \\ \frac{dv_i}{dt} &= -a_i, & \frac{d}{dt}dv_i &= 0. \end{aligned}$$

We obtain immediately that between collisions $\frac{d}{dt}\mathcal{Q} = 0$. Also in reflections in the faces we have monotonicity. For instance in the reflection in the face $\{x_1 = 0\}$ we get

$$\mathcal{Q}^+ - \mathcal{Q}^- = -2\frac{v_1}{a_1} dv_i^2 \geq 0,$$

since the velocity v_1 immediately before the collision is negative, $\mathcal{Q}^\pm$ denote the value of the form immediately before and after the collision.

Thus the system is completely integrable and monotone with respect to this choice of symplectic sectors. Clearly it cannot be strictly monotone, since all Lyapunov exponents are equal to zero. It turns out that there are exactly $\frac{n(n-1)}{2}$ other hyperplanes such that collisions in these hyperplanes are monotone, when they occur on one side of the hyperplane (but not on the other). For a given unit vector $b \in \mathbb{R}^n$ let us consider the hyperplane Π_b perpendicular to b. The direction of b defines the *positive side* of Π_b

Proposition 3.1. *The elastic collision with Π_b on its positive side is $\mathcal{Q}$-monotone if and only if b lies in one of the coordinate planes (i.e., it has only two nonzero components, say b_1 and b_2, is orthogonal to the acceleration vector a and $b_1 + b_2 \geq 0$.*

Proof. In a collision with Π_b the position variables are not changed and the velocities immediately before and immediately after the collision are related by

$$v^+ = v - 2\langle v, b\rangle b.$$

The same relation applies to tangent vectors. The first sum in $\mathcal{Q}$ is invariant under any reflection. The second sum yields

$$\mathcal{Q}^+ - \mathcal{Q}^- = \sum_i \frac{1}{a_i}\left(v_i - 2b_i \sum_j b_j v_j\right)\left(dv_i - 2b_i \sum_j b_j dv_j\right)^2 - \sum_i \frac{v_i}{a_i}(dv_i)^2.$$

The question is when is this quadratic form positive semidefinite for all choices of the velocity v (before the collision) such that $\langle v, b\rangle \leq 0$. Taking $v = \pm(b_2, -b_1, 0, \ldots, 0)$ and restricting the form to the subspace $\{dv_3 = \cdots = dv_n = 0\}$ we obtain

$$\pm\frac{b_2}{a_1}\left[\left((1-2b_1^2)dv_1 - 2b_1 b_2 dv_2\right)^2 - (dv_1)^2\right] \mp \frac{b_1}{a_2}\left[\left(2b_1 b_2 dv_1 - (1-2b_2^2)dv_2\right)^2 - (dv_2)^2\right].$$

The resulting form is on one hand changed when we change the sign of v and on the other hand it should stay positive semidefinite. Hence it must vanish. As a consequence either $b_1^2 + b_2^2 = 0$ or $b_1^2 + b_2^2 = 1$ and $a_1 b_1 + a_2 b_2 = 0$. Putting $a_1 = a\cos\gamma, a_2 = a\sin\gamma, b_1 = -\sin\gamma, b_2 = \cos\gamma$ we arrive at

$$\mathcal{Q}^+ - \mathcal{Q}^- = -\frac{4}{a}\cot 2\gamma\langle v,\, b\rangle\langle dv,\, b\rangle^2 \geq 0, \tag{3.1}$$

if only $\gamma \leq \frac{\pi}{4}$.

□

The $\frac{n(n-1)}{2}$ hyperplanes from Proposition 3.1 divide the positive octant into $n!$ simplicial cones. Any simplicial cone will be called a *wedge*. More precisely, given a basis $\{e^1, \ldots, e^n\}$ in the Euclidean space $\mathbb{R}^n$ we define the wedge $W = W(e^1, \ldots, e^n)$ generated by this basis as

$$W = \{x \in \mathbb{R}^n | x = \eta_1 e^1 + \cdots + \eta_n e^n, \eta_i \geq 0, i = 1, \ldots, n\}.$$

Let us consider one of the $n!$ wedges of the splitting

$$W_* = \{x \in \mathbb{R}^n | a_1^{-1}x_1 \leq a_2^{-1}x \leq \cdots \leq a_n^{-1}x_n\}.$$

Every other wedge of the splitting is determined by an ordering of the rescaled coordinates $a_1^{-1}x_1, a_2^{-1}x, \ldots, a_n^{-1}x_n$. The wedge W_* is generated by the basis $\{e^1, \ldots, e^n\}$

$$\begin{aligned} e^1 &= (a_1, a_2, \ldots, a_{n-1}, a_n), & f^1 &= (a_1^{-1}, 0, \ldots, 0, 0),\\ e^2 &= (0, a_2, \ldots, a_{n-1}, a_n), & f^2 &= (-a_1^{-1}, a_2^{-1}, \ldots, 0, 0),\\ & & &\ldots\\ e^n &= (0, 0, \ldots, 0, a_n), & f^n &= (0, 0, \ldots, -a_{n-1}^{-1}, a_n^{-1}),\end{aligned}$$

and $\{f^1, \ldots, f^n\}$ is the dual basis.

The wedges of the splitting are characterized by the property that $\langle f^k,\, f^l\rangle = 0$ except for $|k-l| \leq 2$, and $\langle f^k,\, f^{k+1}\rangle < 0$. We will call such wedges *simple*. An equivalent property is that for any $k = 1, \ldots, n-1$, the orthogonal projection of e^k on the $n-k$ dimensional subspace spanned by $\{e^{k+1}, \ldots, e^n\}$ is a positive multiple of e^{k+1}. Yet another equivalent property for a simple wedge is that

$$\begin{aligned} &\langle e^k, e^k\rangle = 1, \quad \langle e^k, e^{k+1}\rangle > 0, \quad \text{for} \quad k = 1, \ldots, n-1, \qquad \text{and}\\ &\langle e^k, e^l\rangle = \langle e^k, e^{k+1}\rangle\langle e^{k+1}, e^{k+2}\rangle \ldots \langle e^{l-1}, e^l\rangle \quad \text{for all} \quad 1 \leq k \leq l-1 \leq n-1.\end{aligned}$$

Note that these properties involve a distinguished ordering of the generators. It is not hard to see that there is only one distinguished ordering, up to complete reversal. So in a simple wedge we have a pair of special generators, the first generator and the last generator, and we are free to choose which is first and which is last. A more detailed discussion of simple wedges and their properties can be found in [W6].

Finally our Hamiltonian system restricted to the simple wedge W^* is monotone if $a_1 \geq a_2 \geq \cdots \geq a_n$. This can be translated into the language of the geometry of the simple wedge as the condition that for any $k = 1, \dots, n-1$, the angle between $(n-k)$-dimensional subspaces, $span\{e^k, e^{k+2}, \dots, e^n\}$ and $span\{e^{k+1}, e^{k+2}, \dots, e^n\}$, of the $(n-k+1)$-dimensional space $span\{e^k, e^{k+1}, \dots, e^n\}$, is not smaller than $\frac{\pi}{4}$. Note that this last condition does not allow a complete reversal of the distinguished ordering.

It is interesting to extend the calculations above to the case of nonlinear potential $H = \sum_{i=1}^n \frac{v_i^2}{2} + U(x_1, \dots, x_n)$. Although we cannot easily split the hamiltonian as $H = h_1 + \dots + h_n$, the splitting $dH = dh_1 + \dots + dh_n$ can be generalized by defining the forms $dh_i = v_i dv_i + U_i dx_i$, where $U_i = \frac{\partial U}{\partial x_i}$. Again the subspace $L_2 = \{dh_1 = \dots = dh_n = 0\}$ is Lagrangian and the respective quadratic form becomes $\mathcal{Q} = \sum_i dx_i dv_i + \sum_i \frac{v_i}{U_i} dv_i^2$.

Proposition 3.2. *The system is $\mathcal{Q}$-monotone only for the potential $U = \sum_i f_i(x_i)$.*

Proof. We restrict the form $\frac{d\mathcal{Q}}{dt}$ to the subspace $dx_1 = \dots = dx_n = 0$. We get

$$\frac{d\mathcal{Q}}{dt} = -\sum_i \frac{v_i}{U_i^2} \sum_{i,j} U_{ij} v_j (dv_i)^2,$$

where $U_{ij} = \frac{\partial U}{\partial x_i \partial x_j}$. To guarantee that it is a positive semidefinite form for all velocities v we need to have that $U_{ij} = 0$ for $i \neq j$. $\square$

Thus we can play our game only for completely integrable systems. If we want to add elastic collisions to the system to make it strictly monotone, we are further restricted by Proposition 3.1. By straightforward analysis we conclude that we can have $\mathcal{Q}$-monotone collisions only with hyperplanes orthogonal to coordinate planes, e.g. $b_1 x_1 + b_2 x_2 = 0$. Moreover the acceleration vector must lie in the hyperplane which for the above example yields $b_1 f_1'(x_1) + b_2 f_2'(x_2) = 0$. It shows that the potentials f_1 and f_2 must coincide up to scaling.

The conclusion is that in our example we can obtain strict monotonicity only under very stringent algebraic conditions. Two questions arise. How to extend the scheme to allow more general systems ? Are there any physically natural systems that fit into this limited scheme ? We do not know the answer to the first question. The rest of the paper will be devoted to the discussion of systems which give an answer to the second question.

§4 The falling particle in a 2-dimensional wedge.

We shall consider the motion of a point mass in $\mathbb{R}^2$ falling with constant acceleration and bouncing off a floor.

Special cases of this problem are known to be completely integrable or nonuniformly hyperbolic. Painlevè and Chaplygin established complete integrability in the case of a parabolic floor [K-T].

The case of a symmetric wedge floor was studied numerically by Lehtihet and Miller [L-M]. They discovered that angles exceeding 90 degrees lead to the absence of "elliptic islands". It was shown in [W4] that such a system has indeed nonvanishing Lyapunov exponents almost everywhere, without assuming that the wedge is symmetric. Chernov [Ch] and Liverani and Wojtkowski [L-W] showed that the methods of Sinai and Chernov apply to this problem. One obtains that the system is ergodic on every energy level.

Let us recast the case of an arbitrary (not necessarily) symmetric wedge into the scheme of §3. The hamiltonian of our system is

$$H = \frac{1}{2}w_1^2 + \frac{1}{2}w_2^2 + y_2,$$

and the configuration space is $W = \{y_2 \geq \tan\beta(y_1)y_1\}$, Fig.1a, where $\beta(y_1)$ is equal to β_l for $y_1 < 0$ and to β_r for $y_1 > 0$. Note that β_l is negative.

To use the formalism of §3 we cut up the configuration space into two wedges, the *left wedge* $W_l = \{y_2 \geq \tan\beta_l y_1 \geq 0\}$ and the *right wedge* $W_r = \{y_2 \geq \tan\beta_r y_1 \geq 0\}$. We treat each of these wedges as part of the straightangled wedge for which we have the distinguished symplectic sectors as defined in §3. We obtain two families of symplectic sectors defined by the quadratic forms

$$\mathcal{Q}_{r(l)} = dH_{r(l)}dw_1 + dHdw_2,$$

where $H_{r(l)} = y_1 + \cot 2\beta_{r(l)}w_1^2 + w_1w_2$ are first integrals of the continuous time dynamics. We get monotonicity of reflections in the vertical axis if $|\beta_{r(l)}| \leq \frac{\pi}{4}$ (compare with (3.1)). Monotonicity at the crossing from one wedge to the other requires that the form

$$\mathcal{Q}_r - \mathcal{Q}_l = 2\frac{\sin 2(\beta_l - \beta_r)}{\sin 2\beta_r \sin 2\beta_l}w_1(dw_1)^2$$

is nonnegative if w_1 is positive. It will be the case if the wedge is so wide that $\beta_r - \beta_l < \frac{\pi}{2}$. We conclude that indeed the system in a wide wedge has nonzero Lyapunov exponents. Note that the proof of hyperbolicity requires the use of piecewise continuous quadratic forms. In the present context it seems quite natural since the configuration space has the corner at which dynamics cannot be continuously extended. However such a scheme is surprising in a multidimensional system of §5.

We can generalize this result somewhat, allowing the particle to bounce on an arbitrary piecewise linear floor. The conditions of monotonicity are shown in Fig.1b. In particular if we replace each side of a wide wedge with a concave broken line we obtain a hyperbolic system. The question arises whether smooth concave curves also produce hyperbolicity. We do not know the answer to this question. Since the hyperbolicity in the system is only nonuniform even small perturbations (concave smooth curves with very small curvature) could produce elliptic periodic orbits. It would be interesting to resolve this question.

Finally let us consider the integrable case of the wedge with the straight angle, Fig 2a. The integrability of the system follows from the fact that it is a product of two one dimensional systems of particles falling to the floors with, in general different, accelerations depending on the position of the wedge. The energies of these particles are preserved. Fixing them we obtain an invariant torus which projects into the configuration space as a rectangle with two sides on the sides of the wedge and the fourth vertex on the energy level. Thus for every level of the energy, we obtain a one parameter family of invariant tori.

Let us now modify the wedge by braking the sides at arbitrary heights as shown in Fig.2b. The behavior of a falling particle in such a wedge depends very strongly on the energy level. For energy levels below the lower of the breaks we still have only invariant tori (the particle "does not know about the breaks"). For energy levels exceeding the height of the lower break but below the upper break, a hyperbolic component of positive Lebesgue measure appears, and there is a one parameter family of invariant tori. For energy levels exceeding the upper break another hyperbolic component appears, but there is still a package of invariant tori which "do not explore the breaks". At the critical level, depicted in Fig.2b, there is only one invariant torus left, which separates the two hyperbolic components. Finally at higher levels of the energy the two hyperbolic components coalesce into one. In this scenario we expect that the system is ergodic in a hyperbolic region, but it remains unproven for technical reasons. Namely the singularities of the system are not properly aligned, see [L-W] for the presentation of the Sinai-Chernov methods of proving ergodicity and the discussion of the role of proper alignment of singularities. By the structural theory of hyperbolic systems with singularities developed by Katok and Strelcyn [K-S] we conclude that our system has at most countably many ergodic components in the hyperbolic regions.

§5 Systems with arbitrary number of degrees of freedom.

Let us consider the system of falling particles in a line (PFL system) introduced and studied in [W3]. A PFL system is the system of n point particles in the line with positions $0 \leq q_1 \leq q_2 \leq \cdots \leq q_n$ and masses $m_1, ..., m_n$, falling with constant acceleration (equal to 1) towards the floor of infinite mass (at 0). The particles collide elastically with each other and the floor. Hence between collisions the motion of the particles is governed by the hamiltonian $H = \sum_{i=1}^{n} \left(\frac{p_i^2}{2m_i} + m_i q_i \right)$. Clearly in the case of equal masses we obtain a completely integrable system. The configuration space of the system is a simple wedge $W = \{q \in \mathbb{R}^n | 0 \leq q_1 \leq q_2 \leq \cdots \leq q_n\}$ and the acceleration vector is its first generator. Hence we can employ the symplectic sectors of §3. The result is that if the masses are nonincreasing, $m_1 \geq m_2 \geq \cdots \geq m_n$, and are not all equal then the system is partially hyperbolic, see [W3] for the original proof. Simanyi,[S], showed that, if $m_1 > m_2 \geq \cdots \geq m_n$, then the system is completely hyperbolic. These conditions on the masses are equivalent to the geometric conditions on the simple wedge we encountered in §3. Note that every simple wedge appears as the configuration space of an PFL system with appropriate masses, depending on the geometry of the wedge.

The PFL system with equal masses is completely integrable. It was shown in [Ch-W] that for increasing masses linearly stable periodic orbits appear. Hence the conditions for complete hyperbolicity are sharp.

The monotonicity of the system with nonlinear potential as introduced in Proposition 3.1 can be physically realized in PFL systems governed by the hamiltonian $H = \sum_{i=1}^{n} \left(\frac{p_i^2}{2m_i} + m_i U(q_i) \right)$. The condition of strict monotonicity is that $U'' < 0$, see [W4] for more details. Note that by Proposition 3.2 our scheme does not allow any interaction term in the potential.

Let us modify a PFL system so that the configuration space is the same simple wedge but the acceleration vector is parallel to the last generator rather than the first. We obtain the hamiltonian $H = \sum_{i=1}^{n} \frac{p_i^2}{2m_i} + m_n q_n$. We call it the *capped system* of particles because only the top particle is pulled towards the floor all the other particles move with constant velocities until they collide elastically with nearest neighbors or the floor. The top particle acts as a cap on the system, preventing all the other particles from flying away. The capped systems is equivalent to a PFL system with new masses $\widehat{m}_k, k = 1, \ldots, n$. Normalizing $\widehat{m}_n = 1$ and $m_1 + \cdots + m_n = 1$ we obtain for $k = 1, \ldots, n-1$,

$$\widehat{m}_k = \frac{1}{m_{n-k+1} + \cdots + m_n} - \frac{1}{m_{n-k} + m_{n-k+1} + \cdots + m_n}.$$

We can immediately obtain conditions on the masses in the capped system which guarantee complete hyperbolicity ([W6]). Complete integrability occurs when $m_k = k^{-1} - (k+1)^{-1}$ for $k = 1, \ldots, n-1$, and $m_n = n^{-1}$.

Finally we onsider the system with the hamiltonian of the form $H = \sum_{i=0}^{n} \frac{p_i^2}{2m_i} + U(q_0, q_1, \ldots, q_n)$. We assume that the potential function $U(q_1, \ldots, q_n)$ is a linear translationally invariant function. Hence the potential can be written in one of the following equivalent forms.

$$U = \sum_{i<j} a_{ij}(q_j - q_i) = \sum_{i=1}^{n} b_i(q_i - q_{i-1}) = \sum_{i=1}^{n} c_i(q_i - q_0).$$

The translational invariance of the potential allows the Hamiltonian reduction: we fix the total momentum at zero, $p_0 + p_1 \cdots + p_n = 0$, and the center of mass at the origin, $m_0 q_0 + m_1 q_1 \cdots + m_n q_n = 0$. This system is trivial since all the degrees of freedom are decoupled. We have $\frac{d^2 q_i}{dt^2} = const$. Since the accelerations are constant the kinetic energy grows to infinity. The system is not closed. We introduce additional constraints which close the system and couple different degrees of freedom. These constraints amount to making particles impenetrable to some particles while keeping them transparent to others. When two particles arrive at the same spot, and they are not allowed to pass through each other, the elastic collision occurs.

There are two types of constraints which are in a certain sense extreme the "wide" constraints $q_1 - q_0 \geq 0, q_2 - q_0 \geq 0, \ldots, q_n - q_0 \geq 0$, and the "simple" constraints $q_0 \leq q_1 \leq \cdots \leq q_n$.

A convenient interpretation of the system with wide constraints is that of a horizontal floor of finite mass m_0 and n particles of masses $m_1, m_2, \ldots, m_n$. The floor and the particles can move only in the vertical direction and their vertical positions are q_0 and $q_1, \ldots, q_n$, respectively. There is a constant force of attraction between any of the particles and the floor. Moreover the particles collide elastically with the floor and there are no collisions between the particles (they move along different parallel lines perpendicular to the floor). Hence the particles "communicate" with each other only through the collisions with the floor, which is a rather weak interaction.

In the case of simple constraints we can think that the particles move on one vertical line, each particle collides elastically with its nearest neighbors.

In both cases by the Hamiltonian reduction we end up with the system of the form

$$H = \frac{1}{2} \sum_{i,j=1}^{n} k_{ij} \xi_i \xi_j + \sum_{l=1}^{n} c_l \eta_l,$$

where $(\eta, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ are "positions" and "momenta", $K = \{k_{ij}\}$ is a constant positive definite matrix giving the kinetic energy. The constraints translate as the restriction of the system to the configuration wedge

$W = \{\eta = \eta_1 e^1 + \cdots + \eta_n e^n \mid \eta_i \geq 0, i = 1, \ldots, n\}$.

The geometry of the wedge is contained in the quadratic form defined by the kinetic energy. We have that the Gramm matrix $\{\langle e^i, e^j \rangle\} = L$ is equal to the inverse of the matrix $K = \{k_{ij}\}$. When one of the η coordinates vanishes the velocity is changed instantaneously by the rules of elastic collisions, i.e., the component of the velocity parallel to the face of the wedge is preserved and the component orthogonal to the face is reversed.

In the case of simple constraints the configuration wedge is simple. It is reflected by the fact that the matrix $K = L^{-1}$ is tridiagonal and the entries on the superdiagonal are negative (compare with the definition of the simple wedge in terms of the dual basis). If we assume that the constants c_k are proportional to the masses m_k, the acceleration vector has the direction of the firat generator of the wedge. Hence such a system is equivalent to a PFL system with masses $\widehat{m}_k, k = 1, \ldots, n$, different from the current masses. With the normalizations $\widehat{m}_n = 1$ and $m_0 + \cdots + m_n = m_n(m_0 + \cdots + m_{n-1})^{-1}$ we obtain for $k = 1, \ldots, n-1$,

$$\widehat{m}_k = \frac{1}{m_0 + \cdots + m_{k-1}} - \frac{1}{m_0 + \cdots + m_{k-1} + m_k}$$

Again we obtain immediately conditions of complete hyperbolicity ([W6]). The system is completely integrable, if for some constant $a > n$ we have $m_0 = a^{-1}$ and

$m_k = (a-k)^{-1} - (a-k+1)^{-1}, \quad k = 1, \ldots, n.$

It is interesting that such a formula for the masses was also obtained by A.Yu. Plakhov and A.M. Stepin [P-S], in their study of simple scattering in one dimension.

A wedge is called wide, if all the off-diagonal entries of the matrix L are negative. In other words a wedge is wide, if the angles between any pair of generating rays exceed $\frac{\pi}{2}$. In the case of wide constraints the configuration wedge is wide.

Main Theorem [W6]. *The system of a particle in a wide wedge with arbitrary acceleration vector from the interior of the dual wedge is completely hyperbolic.*

In particular the system of particles falling to a floor of finite mass is completely hyperbolic without any assumptions on the masses. The Main Theorem is a multidimensional generalization of the two dimensional particle in a wide wedge from §4.

The striking feature of the proof of the Main Theorem is that we split the wide wedge into $n!$ simple wedges with the acceleration vector as the common first generator of all these wedges. When the particle moves in one simple wedge we use the symplectic sectors furnished by the construction of §3 (or the isomorphism with the appropriate PFL system). When the orbit crosses to the adjacent simple wedge we consider another form $\mathcal{Q}$. Hence the family of quadratic forms used to detect hyperbolicity is discontinuous. Let us note that the partition into simple wedges depends on the direction of the acceleration vector and unless $c_k \sim m_k$, it does not have a clear meaning in the language of the particles falling to a floor of finite mass. In the special (and natural) case where $c_k \sim m_k$ this splitting of W is just the splitting according to the ordering of the variables $\eta_k = q_k - q_0$.

This method can be extended to the case of constraints intermediate between the wide and the simple, for example when we stack groups of particles on one line. Assuming that the acceleration of all particles in one stack is the same we obtain sufficient conditions for complete hyperbolicity ([W6]). As we add more constraints the conditions of complete hyperbolicity become stricter and in the case of one stack we can have complete integrability as previously described.

This paradoxical behavior, that with weaker interactions we have equipartition of energy, which may fail for stronger interactions, can be understood by the following observation. Let us modify the original wide system of noninteracting particles falling to the floor by splitting each mass into two or more masses that are stacked on one vertical line. In the original system the particles have to freely "share" their energy with the floor and hence with other particles. In the modified system the stack of particles acts as "internal" degrees of freedom which may store energy for extended periods of time. One would expect that the energy transfer between stacks is less vigorous than in the case when all the masses in one stack are glued into one particle.

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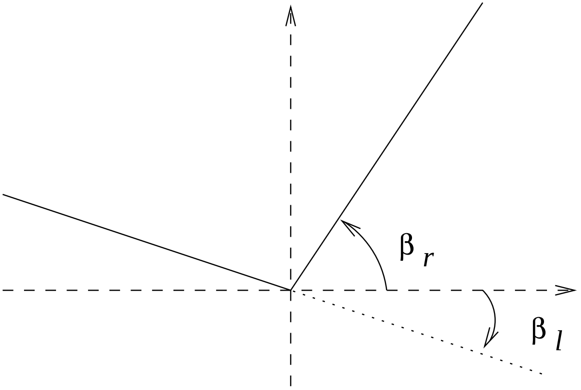

Fig.1a

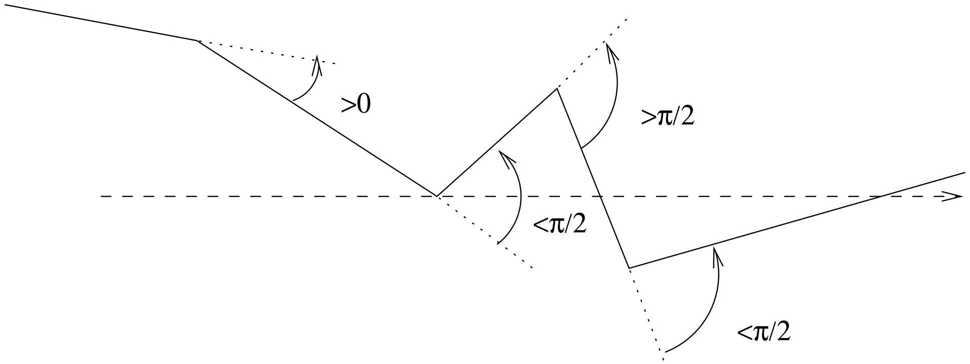

Fig.1b

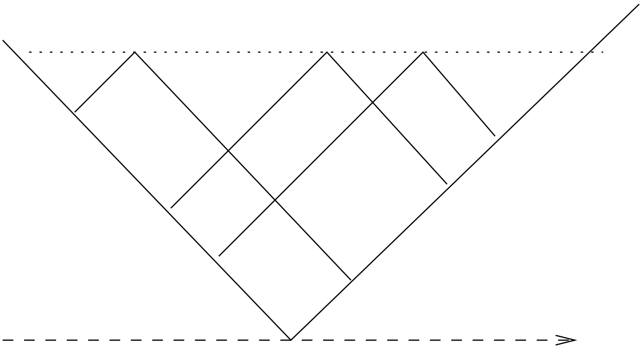

Fig.2a

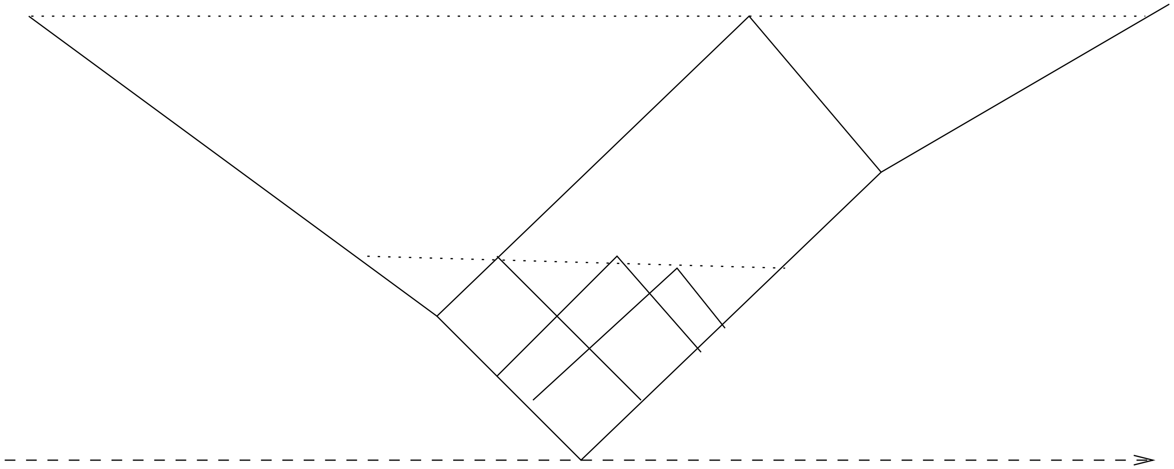

Fig.2b Critical energy level and the last invariant torus