HYPERBOLIC BILLIARDS

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1. Introduction

Billiards are a class of dynamical systems with appealingly simple description. A point particle moves with constant velocity in a box of arbitrary dimension (the billiard table) and reflects elastically from the boundary (the component of velocity perpendicular to the boundary is reversed and the parallel component is preserved). Mathematically it is a class of hamiltonian systems with collisions defined by symplectic maps on the boundary of the phase space. The billiard dynamics defines a one parameter group of maps Φ^t of the phase space which preserve the Lebesgue measure, and are in general only measurable due to discontinuities. The boundaries of the box are made up of pieces, concave, convex and flat. Discontinuities occur at the orbits tangent to concave pieces of the boundary of the box. The orbits hitting two adjacent pieces (”corners”) cannot be naturally continued, which is another source of discontinuities. These singularities are not too severe so that the flow has well defined Lyapunov exponents and Pesin structural theory is applicable, [K-S]. A billiard system is called hyperbolic if it has nonzero Lyapunov exponents on a subset of positive Lebesgue measure, and completely hyperbolic if all of its Lyapunov exponents are nonzero almost everywhere, except for one zero exponent in the direction of the flow.

Billiards in smooth strictly convex domains have no singularities, but no such examples are known to be hyperbolic.

In general billiards exhibit mixed behavior just like other hamiltonian systems, there are invariant tori intertwined with ”chaotic sea”. In hyperbolic billiards stable behavior is excluded by the choice of the pieces in the boundary of the box, arbitrary concave pieces and special convex ones, and their particular placement. Thus hyperbolicity is achieved by design, as in optical instruments.

It is was established by Turaev and Rom-Kedar that complete hyperbolicity is lost under generic singular perturbation of the billiard system to a smooth hamiltonian system, [T-RK].

Hyperbolicity is the universal mechanism for random behavior in deterministic dynamical systems. Under suitable additional assumptions it leads to ergodicity, mixing, K-property, Bernoulli property, decay of correlations, central limit theorem, and other stochastic properties. Hyperbolic billiards provide a natural class of examples for which these properties were studied. In this article we restrict ourselves to hyperbolicity itself.

The most prominent example of a hyperbolic billiard is the gas of hard spheres. This way of looking at the system was developed in the groundbreaking papers of Sinai, see [Ch-S] for an exhaustive list of references. The collection of papers, [H], contains more up to date information. Another source on hyperbolic billiards is the book by Chernov and Markarian, [Ch-M]. The books by Kozlov and Treschev [K-T], and by Tabachnikov [T] provide broad surveys of billiards from different perspectives.

2. Jacobi fields and monotonicity

The key to understanding hyperbolicity in billiards lies in two essentially equivalent descriptions of infinitesimal families of trajectories. The basic notion is that of a Jacobi field along a billiard trajectory. Let $\gamma(t,u)$ be a family of billiard trajectories, where t is time and u is a parameter, $|u| < \epsilon$. A Jacobi field along $\gamma(t,0)$ is defined by $J(t) = \frac{\partial \gamma}{\partial u}_{|u=0}$.

Jacobi fields form a finite dimensional vector space which can be identified with the tangent to the phase space at points along the trajectory. They contain the same information as the derivatives of the billiard flow $D\Phi^t$. In particular the Lyapunov exponents are the exponential rates of growth of Jacobi fields.

Jacobi fields split naturally into parallel and perpendicular components to the trajectory, each of them a Jacobi field in its own right. The parallel Jacobi field carries the zero Lyapunov exponent. In the rest we discuss only the perpendicular Jacobi fields. Between collisions the Jacobi fields satisfy the differential equation $J" = 0$, hence $J(t) = J(0) + tJ'(0)$. At a collision a Jacobi field undergoes a change by the map

$$J(t_c^+) = \mathcal{R}J(t_c^-) \qquad J'(t_c^+) = \mathcal{R}J'(t_c^-) + \mathcal{P}^*\mathcal{K}\mathcal{P}J(t_c^+), \tag{1}$$

where $J(t_c^-)$ and $J(t_c^+)$ are Jacobi fields immediately before and after collision, $\mathcal{K}$ is the shape operator of the piece of the boundary ($\mathcal{K} = \nabla n$, n is the inside unit normal to the boundary), and $\mathcal{P}$ is the projection along the velocity vector from the hyperplane perpendicular to the orbit to the hyperplane tangent to the boundary. Finally $\mathcal{R}$ is the orthogonal reflection in the hyperplane tangent to the boundary.

Perpendicular Jacobi fields at a point of a trajectory can be identified with a subspace of the tangent to the phase space, the subspace perpendicular to the phase trajectory. To measure the growth/decay of Jacobi fields we introduce a quadratic form on the tangent spaces, or equivalently on Jacobi fields, $\mathcal{Q}(J,J') =< J, J' >$. Evaluation of $\mathcal{Q}$ on a Jacobi field is a function of time $\mathcal{Q}(t)$. Between collisions we have $\mathcal{Q}(t_2) \geq \mathcal{Q}(t_1)$ for $t_2 \geq t_1$ (*monotonicity*). By (1) the monotonicity at the collisions, i.e., $\mathcal{Q}(t_c^+) \geq \mathcal{Q}(t_c^-)$ is equivalent to the positive semidefinitness of the shape operator $\mathcal{K} \geq 0$, it holds for concave pieces of the boundary. If $\mathcal{K} > 0$ at a point of collision with the boundary, then for $(J,J') \neq (0,0)$, we have $\mathcal{Q}(t_2) > \mathcal{Q}(t_1)$ (*strict monotonicity*), assuming that the collision occurred between time t_1 and t_2.

In billiards with concave pieces of the boundary, where $\mathcal{K} \geq 0, \mathcal{K} \neq 0$, strict monotonicity may still occur after sufficiently many reflections (*eventual strict monotonicity*, or *ESM*). Such billiards are called semidispersing, and the gas of hard spheres is an example.

The role of monotonicity is revealed in the following

Theorem [W1]**.** *If a system is eventually strictly monotone (ESM), except on a set of orbits of zero measure, then it has nonzero Lyapunov exponents almost everywhere except for one zero exponent (carried by the parallel Jacobi field).*

The theorem applies to billiard systems. It can be generalized and applied to other systems, not even hamiltonian (see [W2] for precise formulations, references and the history of this idea).

The difficulty in applying the Theorem to the gas of hard spheres lies in the gap between monotonicity and strict monotonicity. There are many orbits on which strict monotonicity is never attained (*parabolic orbits*). Establishing that the family of parabolic orbits has measure zero (or better yet codimension 2) is a formidable task. It was brought to conclusion in the work of Simányi, [S].

3. Wave fronts and monotonicity There is a geometric formulation of monotonicity (which historically preceded the one given above). Let us consider a local wavefront, i.e., a local hypersurface $W(0)$ perpendicular to a trajectory $\gamma(t)$ at $t = 0$. Let us consider further all billiard trajectories perpendicular to $W(0)$. The points on these trajectories at time t form a local hypersurface $W(t)$ perpendicular again to the trajectory (warning: at exceptional moments of time the wavefront $W(t)$ may be singular). Infinitesimally wavefronts are described by the shape operator $U = \nabla n$, where n is the unit normal field. U is a symmetric operator on the hyperplane tangent to the wavefront (and perpendicular to the trajectory $\gamma(t)$. The evolution of infinitesimal wavefronts is described by the formulas

$$\begin{aligned} U(t) &= (t + U(0)^{-1})^{-1} \qquad \text{without collisions} \\ U(t_c^+) &= \mathcal{R}U(t_c^-)\mathcal{R}^{-1} + \mathcal{P}^*\mathcal{K}\mathcal{P} \qquad \text{at a collision} \end{aligned} \tag{2}$$

It follows that between collisions a wavefront that is initially convex (i.e., diverging, or $U > 0$) will stay convex. Moreover any wavefront after a sufficiently long run without collisions will become convex (after which the normal curvatures of the wavefront will be decaying). The second part of (2) shows that after a reflection in a strictly concave boundary a convex wavefront becomes strictly convex (and its normal curvatures increase). These properties are equivalent to (strict) monotonicity as formulated above. Indeed in the language of Jacobi fields an infinitesimal wavefront represents a linear subspace in the space of perpendicular Jacobi fields, i.e., the tangent space. (Furthermore it is a Lagrangian subspace with respect to the standard symplectic form.) We can follow individual Jacobi fields or whole subspaces of them. It explains the parallel of (1) and (2). The form $\mathcal{Q}$ allows the introduction of positive and negative Jacobi fields and positive and negative Lagrangian subspaces. An infinitesimal convex wavefront represents a positive Lagrangian subspace. Monotonicity is equivalent to the property that a positive Lagrangian subspace stays positive under the dynamics (it may appear that there is a loss of information in formulas (2) compared to (1), but actually they are equivalent due to the symplectic nature of the dynamics, [W1]).

4. Design of hyperbolic billiards

In view of (2) it seems that a convex piece in the boundary ($K < 0$) excludes monotonicity. There are two ways around this obstacle. First we could change the quadratic form $\mathcal{Q}$ at the convex boundary. Second we can treat convex pieces as "black boxes" and look only at incoming and outgoing trajectories. Although the second strategy seems more restrictive all the examples constructed to date fit the black box scenario, and we will present it in more detail.

To understand this approach let us consider a billiard table with flat pieces of the boundary and exactly one convex piece. A trajectory in such a billiard experiences visits to the convex piece separated by arbitrary long sequences of reflections in flat pieces, which do not affect the geometry of a wavefront at all. Hence whatever is the geometry of a wavefront emerging from the curved piece it will become convex and very flat by the time it comes back to the curved piece of the boundary again. Hence it follows, at least heuristically, that we must study the complete passage through the convex piece of the boundary, regarding its effect on convex, and especially flat, wavefronts.

Important difference between convex and concave pieces is that a trajectory has usually several consecutive reflections in the same convex piece, moreover the number of such reflections is unbounded. A finite billiard trajectory is called complete if it contains reflections in one and the same piece of the boundary, and it is preceded and followed by reflections in other pieces.

Definition. *A complete trajectory is (strictly) z-monotone if for every nonzero Jacobi field the value of the form $\mathcal{Q}$ (increases) does not decrease between the point at the distance z before the first reflection and the point at the distance z after the last reflection.*

A complete trajectory is parabolic if there is a nonzero Jacobi field J such that J' vanishes before the first and after the last reflection.

In the language of wavefronts a complete trajectory is z-monotone if every diverging wavefront at a distance at least z from the first reflection becomes diverging after the last reflection at the distance z, or earlier.

It turns out that the only obstruction to monotonicity of complete trajectories is parabolicity. More precisely if a complete trajectory is not parabolic then it is z-monotone for some $z > 0$.

It follows from the Theorem that we get a completely hyperbolic billiard if we put together curved pieces with no complete parabolic trajectories and some flat pieces, in such a way that for every two consecutive complete trajectories, being z_1 and z_2-monotone respectively, the distance from the last reflection in the first trajectory to the first reflection in the second one is bigger than $z_1 + z_2$. Indeed we can put together the midpoints of trajectories leaving one curved piece and hitting another one into the Poincare section of the billiard flow and we obtain immediately ESM for the return map.

We can formulate somewhat informally two principles for the design of hyperbolic billiards.

No parabolic trajectories: Convex pieces of the boundary cannot have complete parabolic trajectories.

Separation: There must be enough separation (in space, or in time through reflections in flat pieces) between strictly z-monotone trajectories according to the values of z.

All of the examples of hyperbolic billiards constructed up to now are designed according to these principles.

5. Hyperbolic billiards in dimension 2

Checking the absence of parabolic trajectories is nontrivial due to the unbounded number of reflections in complete trajectories close to tangency. It was accomplished so far only in integrable, or near integrable examples, with the exception of convex scattering pieces described in the following.

Billiards in dimension 2 are understood best. First of all there is yet another way of describing infinitesimal families of nearby trajectories. Every infinitesimal family of rays in the plane has a point of focusing (in linear approximation), possibly at infinity. This point of focusing contains the same information as the curvature of a wavefront (it is the center of curvature, rather than curvature itself) and it has the advantage that it does not change between collisions. The change in the focusing point after a reflection is described by the familiar mirror equation of the geometric optics

$$-\frac{1}{f_0} + \frac{1}{f_1} = \frac{2}{d},$$

where f_0, f_1 are the signed distances of the points of focusing to the reflection point, $d = r\cos\theta$, r is the radius of curvature of the boundary piece ($r > 0$ for a strictly convex piece) and θ is the angle of incidence. The mirror equation is just the two dimensional version of (2).

It is instructive to consider an arc of a circle. Billiard in a disk is integrable due to its rotational symmetry. Let J be a Jacobi field obtained by rotation of a trajectory. This family of trajectories ("the rotational family") is focused exactly in the middle between two consecutive reflections (that is where J vanishes). It follows further from the mirror equation that a parallel family of orbits is focused at a distance $\frac{d}{2}$ after the reflection, and any family focusing somewhere between the parallel family and the rotational family will focus at a distance somewhere between $\frac{d}{2}$ and d, not only after the first reflection, but also after arbitrary long sequence of reflections .

Hence any complete trajectory in an arc of a circle is z-monotone, where $2z$ is the length of a single segment of the trajectory and strictly z'-monotone for any $z' > z$. Two arcs of a circle separated by parallel segments form the stadium of Bunimovich, [B1].

Lazutkin showed that billiards in smooth strictly convex domains are near integrable near the boundary, [L]. Donnay, [D], applied Lazutkin's coordinates to establish that for an arbitrary strictly convex arc the situation near the boundary is similar to that in a circle, i.e., complete trajectories near tangency are z-monotone, where z is of the order of the length of a single segment. In particular no near tangent complete trajectory can be parabolic. Hence this crucial calculation shows that if a strictly convex arc has no parabolic trajectories then any sufficiently small perturbation also has no parabolic trajectories. It follows further that any sufficiently small piece of a given strictly convex arc has no parabolic trajectories.

It turns out that in dimension 2 complete parabolic trajectories are also z-monotone for some $z > 0$ (but clearly not strictly monotone), [W3]. However they are still an obstacle to complete hyperbolicity because generically nearby complete trajectories are z-monotone without a bound for the values of z, so that no separation of convex pieces is sufficient.

Integrability of the elliptic billiard allows one to establish strict monotonicity of trajectories in the semi-ellipse with endpoints on the longer axis, [W4]. Donnay, [D], showed that also the semi-ellipse with endpoints on the shorter axis has no parabolic trajectories provided that the eccentricity is less than $\frac{\sqrt{2}}{2}$. As the eccentricity goes to $\frac{\sqrt{2}}{2}$ the separation required to produce a hyperbolic billiard goes to infinity. Markarian et al, [M-O-P], obtained explicitly the separation of the elliptic pieces needed for hyperbolicity, when the eccentricity is smaller than $\frac{\sqrt{2-\sqrt{2}}}{2}$.

It follows from the mirror equation that a trajectory with one reflection in a convex piece is always strictly z-monotone for $z > d$. Hence if for any two consecutive reflections in convex pieces with respective values of d equal to d_1 and d_2, the distance between reflections exceeds $d_1 + d_2$, then the billiard is completely hyperbolic. For one convex piece this condition, called convex scattering, turns out to be equivalent to $\frac{d^2r}{ds^2} < 0$, where s is the arc length, [W4]. This leads to examples of hyperbolic billiards with one convex piece of the boundary, like the domain bounded by the cardioid.

Also any complete trajectory in a convex scattering piece is strictly z-monotone for z bigger than the maximum of the values of d for the first and the last segment of the trajectory. This allows to find easily the explicit separation of convex scattering pieces guaranteeing hyperbolicity.

6. Hyperbolic in higher dimensions

In higher dimensions only two constructions of hyperbolic billiards with convex pieces in the boundary are known. The first construction by Bunimovich, [B2], involves a piece of a sphere whose angular size, as seen from the center, does not exceed $\frac{\pi}{2}$, [W5],[B-R], [W3]. The second construction by Papenbrock, [P], uses two cylinders, at 90 degrees with respect to each other to destroy integrability, [B-De],[W3]. In both cases the successful treatment is based on complete integrability of the billiard systems bounded by a sphere or a cylinder.

In both of these constructions trajectories need to be cut into strictly monotone pieces of unbounded lengths. In the case of spherical caps complete trajectories are z-monotone with unbounded value of z and the geometry of the billiard table is used to separate them in time by sufficiently many reflections in flat pieces of the boundary, [W3]. In the case of cylinders trajectories are cut by consecutive returns to a Poincare section in the middle of the billiard table.

7. Soft billiards

The same ideas of monotonicity and strict monotonicity are applicable to soft billiards where specular reflections are replaced by scatterers in which the point particle is subjected to the action of a spherically symmetric potential. As in ordinary billiards we compare the wave fronts along trajectories before entering and after leaving scatterers. Again in the absence of parabolic trajectories sufficient separation of the scatterers produces a completely hyperbolic system.

The conditions on the potential that guarantee the absence of parabolic trajectories were obtained by Donnay and Liverani, [D-L], in the two dimensional case and by Bálint and Tóth, [B-T], in higher dimensions. The complete integrability of the motion of a point particle in a spherically symmetric potential is crucial in the derivation of these conditions, [W3].

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