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Maciej P. Wojtkowski, Department of Mathematics, University of Arizona, Tucson, Arizona 85 721 USA

E-mail address: maciejw@math.arizona.edu

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Assuming that the obstacles are Weyl convex we obtain hyperbolic properties of the billiard W–flows, [W00], in parallel with Section 4.

Two dimensional Lorentz gas with round scatterers of radius r, a constant electric field E and the Gaussian thermostat is a model of this kind. It was studied numerically by Moran and Hoover, [MH87]. Chernov et al,[CELS92], obtained exaustive rigorous results about its SRB measures in the case of small fields and finite horizon. We can prove the uniform hyperbolicity of the model when $r|E| < 1$ and this inequality is sharp. Indeed, the exponential map $F(z) = e^{|E|z}$, takes the trajectories of the W–flow onto the straight lines and being conformal it respects the reflections from the boundary. Hence the mapping F takes the billiard W–flow into a billiard flow, at least locally (globally we get billiards on the Riemann surface of the logarithm). However the image of a disk of radius r under the exponential mapping is strictly convex if and only if $r|E| < 1$. Once the obstacles loose convexity we readily find elliptic periodic orbits which rules out global hyperbolicity, [W00].

C. For obscure reasons in hamiltonian systems of many particles interacting by a pair potential, which are expected to have in general good statistical properties (and do have them in numerical experiments), hyperbolicity in all of the phase space is rarely encountered. The notable exceptions are the Boltzmann-Sinai gas of hard spheres,[S00], and also the one dimensional systems of falling balls, [W98],[W99]. In particular, beyond the 2 dimensional examples of Knauf, [K88],(which have Weyl counterparts, [W00]), we do not know of systems equivalent to geodesic flows on manifolds with negative sectional curvatures.

For systems of particles in an external field the Gaussian thermostat provides additional interactions and the resulting system is not hamiltonian. Examining the simplest examples we find that also in this case the zero and positive Weyl sectional curvatures are common.

For noninteracting particles we get a cartesian product of Weyl manifolds and hence we get zero sectional curvature in some directions, as in A.

When we consider the system of two elastic disks in the 2-dimensional torus, the cylinders that are cut out from the four dimensional configuration space are Weyl convex, but the zero Weyl curvature is enough to allow the presence of local simple attractors of the type of Example 1.2, which rules out global fast mixing and decay of correlations.

For the Lorentz gas of two noninteracting point particles in a 2 dimensional torus with round scaterrers, an external field and the Gaussian thermostat, the calculation of (5.6) shows that the obstacles (products of disks) in the four dimensional torus of configurations are not Weyl convex everywhere.

These examples indicate that in the setup of Weyl geometry there is no more freedom for the occurence of global hyperbolicity then in the riemannian realm. The difficulty in constructing natural examples satisfying the Chaotic Hypothesis of Gallavotti,[G01], seems to be parallel to the scarcity of multidimensional hamiltonian systems with strong mixing properties, [L00].

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with negative riemannian curvature. The following two observations suggest that it is not necessarily so.

If $(M_i, g_i, E_i), i = 1, 2$ are two Weyl manifolds then their cartesian product has a natural Weyl structure, and the Weyl sectional curvature in the direction of the plane spanned by $(E_1, 0)$ and $(0, E_2)$ is either mixed (positive, negative and zero) or always zero (iff $|E_i| = const, i = 1, 2$). Hence just like in the riemannian case, product manifolds cannot have negative sectional curvature.

Secondly, if we look for interesting homogeneous Weyl structures we are confronted with the phenomenon that on symmetric spaces of noncompact type, [H78], there are no homogeneous Weyl structures at all (except for the riemannian metric itself). The only simply connected homogeneous riemannian manifolds with a compact factor and nonpositive sectional curvature are symmetric spaces,[AW76].

Conjecture 5.4. *The only simply connected homogeneous Weyl manifolds with a compact factor and negative Weyl sectional curvatures are riemannian symmetric spaces.*

We obtain a homogeneous Weyl manifold by taking a left invariant metric g on a Lie group and a left invariant vector field E. It is known that on unimodular Lie groups the only left invariant metrics with nonpositive riemannian sectional curvature are metrics with zero curvature, [AW76],[M76]. In contrast, by (3.5) the n-dimensional torus, $n \geq 3$, with a constant vector field E has negative Weyl sectional curvature in the direction of any plane except for the planes that contain E, where it vanishes.

Another instructive example is the 3-dimensional Lie group SOL,[T98], which has mixed riemannian sectional curvatures, but for one of the left invariant fields the Weyl sectional curvature is nonpositive, with some negative curvature.

A weaker version of Conjecture 5.4 is

Conjecture 5.5. *There are no left invariant Weyl structures with negative Weyl sectional curvatures on unimodular Lie groups.*

B. Billiard systems are a natural extension of geodesic flows. Similarly we can consider billard W–flows on Weyl manifolds with boundaries by augmenting the continuous time dynamics with elastic reflections in the boundaries. For example we can remove from a Weyl manifold some subsets ("obstacles"). In case of convex obstacles in a cube (or a flat torus), we obtain Sinai billiards, which have good statistical properties.

We can introduce Weyl strict convexity of the obstacles by requiring that the Weyl geodesics in the exterior of the obstacle can have locally at most one point in common with the obstacle. It can be calculated that this property has the following infinitesimal description. Let N be the field of unit vectors orthogonal to the obstacle and pointing out of it. The riemannian convexity of the obstacle at a point is defined by the positive definiteness of the riemannian shape operator $K\xi_0 = \nabla_{\xi_0} N$, where ξ_0 is from the tangent subspace to the obstacle. Similarly we introduce the operator

$$\widehat{K}\xi_0 = K\xi_0 + \langle N, E\rangle \xi_0, \tag{5.6}$$

which is the orthogonal projection of the " Weyl shape operator" $\widehat{\nabla}_{\xi_0} N$ to the tangent subspace. An obstacle is (strictly) Weyl convex if $\widehat{K}$ is positive (definite) semidefinite.

Corollary 4.6. *If the local potential is harmonic and the Gaussian curvature $K < 0$ on M then the W–flow is a transitive Anosov flow.*

We conclude that in the case of fields given by automorphic forms on surfaces of constant negative curvature, which were studied by Bonetto, Gentile and Mastropietro, [BGM00], the flow is always Anosov. Further it follows from the theory of SRB measures that if such a flow is Anosov then it is also automatically dissipative, i.e., the SRB measure is singular, [W00'].

Note that in this situation we can multiply the vector field E by an arbitrary scalar λ and we still get a transitive Anosov flow. It would be interesting to understand the asymptotics of the SRB measure as $\lambda \to \infty$. Is the limit supported on the union of the integral curves of E? Let us stress that this scenario differs from the perturbative conditions in [Go97], [Gr99], [W00'], where the geodesic curvature of the trajectories cannot be too large. Our trajectories may have arbitrarily large geodesic curvatures and yet they form a transitive Anosov flow.

5. Examples, extensions, open problems and disappointments.

A. In view of Theorems 4.2 and 4.4 it is natural to ask, if the negative sectional Weyl curvature is enough to guarantee the Anosov property for the W–flow. It follows immediately from (4.1) and (3.5) that if for every plane Π the sectional Weyl curvatures satisfy

$$\widehat{K}(\Pi) + \frac{1}{4}(E^2 - E_\perp^2) = K(\Pi) - div_\Pi E + \frac{1}{4}E^2 - \frac{5}{4}E_\perp^2 < 0,$$

then the W–flow is Anosov. We propose

Conjecture 5.1. *There are manifolds of dimension ≥ 3 and tangent vector fields E such that the Weyl sectional curvatures are negative everywhere but the W–flow is not Anosov.*

It is also plausible that under the assumption of negative sectional curvatures we can obtain W–flows which are nontransitive Anosov flows, as in the examples of Franks and Williams, [FW80].

To resolve these issues we would like to construct examples of Weyl manifolds with negative sectional curvatures which are not small deformations of riemannian metrics of negative sectional curvature. In that direction we found some obstructions.

Proposition 5.2. *There are no Weyl structures with negative sectional curvatures in a small neighborhood of the homogeneous Weyl structure on an n-dimensional torus (as in Example 1.2).*

Conjecture 5.3. *There are no Weyl structures of negative sectional curvature on n-dimensional tori.*

It is so for $n = 2$ since for 2-dimensional manifolds the Weyl curvature is equal to the gaussian curvature of the Gauduchon gauge.

The presence of a negative term in the formula for the Weyl sectional curvature (3.5) gives the impression that it is easier to find manifolds with negative Weyl curvature than

4. Hyperbolic properties of W–flows.

We obtain the information about hyperbolicity of W–flows studying the quadratic form $\mathcal{J}$ in the tangent spaces of the phase space SM, defined by $\mathcal{J}(\xi,\chi) = \langle \xi, \chi \rangle$. The form $\mathcal{J}$ factors naturally to the quotient bundle (the quotient by the span of the vector field (2.1), i.e., in the (ξ,χ) coordinates the quotient by the span of $(v,0)$). The quotient space can be represented by the subspace $\langle \xi, v \rangle = 0$, but this subspace is not invariant under the linearization (3.4) of the flow. The form $\mathcal{J}$ in the quotient space is nondegenerate and it has equal positive and negative indices of inertia.

We take the Lie derivative of $\mathcal{J}$ and obtain

$$\frac{d}{dt}\mathcal{J} = \chi^2 \ - \varphi(v)\mathcal{J} - \widehat{K}(\Pi)\xi^2\,, \tag{4.1}$$

where Π is the plane spanned by v and ξ. Because of the middle term, which is absent in the riemannian case, the negativity of the sectional curvature does not guarantee that (4.1) is positive definite. However it does have a weaker property that it is positive where $\mathcal{J} = 0$. We call a flow with this property *strictly $\mathcal{J}$-separated*,[W01]. A strictly $\mathcal{J}$-separated flow has a dominated splitting, [M̃84], i.e., it has a continuous splitting into "weakly stable" and "weakly unstable" subspaces on which the rates of growth are uniformly separated, but which are not necessarily decay and growth respectively. For example the dominated splitting allows exponential growth in the "weakly stable" subspace, but then all of the "weakly unstable" subspace grows with a bigger exponent. It needs to be stressed that the splitting is done in the quotient space, because in contrast to the riemannian/contact case we do not have a priori any invariant subspaces transversal to the flow direction.

It turns out that negative sectional Weyl curvatures guarantee even more hyperbolicity.

Theorem 4.2. *If the sectional curvatures of the Weyl structure are negative everywhere in M then the W–flow is strictly $\mathcal{J}$-separated and hence it has the dominated splitting into the invariant subspaces $\mathcal{E}^+$ and $\mathcal{E}^-$. Moreover there is uniform exponential growth of volume on $\mathcal{E}^+$ and uniform exponential decay of volume on $\mathcal{E}^-$.*

Corollary 4.3. *If the sectional curvatures of the Weyl structure are negative everywhere in M then for any ergodic invariant measure of the W–flow the largest Lyapunov exponent is positive and the smallest Lyapunov exponent is negative.*

We can apply this corollary to an individual periodic orbit and we obtain linear instability. Moreover there are also no repelling periodic orbits under the assumption of negative sectional Weyl curvature.

The 2-dimensional case is special. We have

Theorem 4.4. *For a 2-dimensional compact surface M if the curvature of the Weyl structure is negative, i.e, $\widehat{K} = K - divE < 0$ on M, then the W–flow is a transitive Anosov flow.*

For a locally potential vector field $divE = -\triangle U$ and we get

Corollary 4.5. *If $K < -\triangle U$ on a 2-dimensional surface M then the W–flow is a transitive Anosov flow.*

of M along the trajectory where we linearize the W–flow. The Weyl parallel transport along a path is a conformal linear mapping and the coefficient of dilation is equal to $e^{-\int \varphi}$. We choose an orthonormal frame $v, e_1, \ldots, e_{n-1}$ in an initial tangent space $T_{q_0}M$ and parallel transport it along a trajectory of our W–flow in the direction $v \in SM$. We obtain the orthogonal frames which we normalize by the coefficient $e^{\int \varphi}$ and denote them by $v(t), e_1(t), \ldots, e_{n-1}(t)$. Let $(\xi_0, \xi_1, \ldots, \xi_{n-1})$ and $(0, \chi_1, \ldots, \chi_{n-1})$ be the components of ξ and χ respectively in these frames. Let further $\widetilde{\xi} = (\xi_1, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1}$ and $\widetilde{\chi} = (\chi_1, \ldots, \chi_{n-1}) \in \mathbb{R}^{n-1}$. The equations (3.3) will read then

$$\frac{d\xi_0}{dt} = \varphi(\xi - \langle \xi, v\rangle v), \quad \frac{d\widetilde{\xi}}{dt} = -\varphi(v)\widetilde{\xi} + \widetilde{\chi}, \quad \frac{d\widetilde{\chi}}{dt} = -\mathcal{R}\widetilde{\xi} \tag{3.4}$$

where the operator $\mathcal{R}\widetilde{\xi} = \widehat{R}_a(\xi, v)v \in \mathbb{R}^{n-1}$ (the vector $\widehat{R}_a(\xi, v)v$, being orthogonal to v, is considered as an element in $\mathbb{R}^{n-1}$ by the expansion in the basis $e_1(t), \ldots, e_{n-1}(t)$). Note that $\widehat{R}_a(\xi, v)v$ and $\varphi(\xi - \langle \xi, v\rangle v)$ depend $\widetilde{\xi} \in \mathbb{R}^{n-1}$ alone.

The linearized equations (3.4) for $(\widetilde{\xi}, \widetilde{\chi})$ differ from the Jacobi equations in the riemannian case by the presence of the "nonconservative" term $-\varphi(v)\widetilde{\xi}$ in the first equation and the properties of the operator $\mathcal{R}$, which although defined analogously in terms of the curvature tensor is not in general symmetric. The curvature tensor can be calculated directly, [W00], but the result is somewhat cumbersome. However the sectional curvatures $\widehat{K}(\Pi)$ of the Weyl connection in the direction of a plane Π defined as

$$\widehat{K}(\Pi) = \langle \widehat{R}_a(X, Y)Y, X\rangle,$$

for any orthonormal basis $\{X, Y\}$ of Π, are pleasantly transparent

$$\widehat{K}(\Pi) = K(\Pi) - E_\perp^2 - div_\Pi E, \tag{3.5}$$

where $K(\Pi)$ is the riemannian sectional curvature in the direction of Π, the vector field $E_\perp$ is the component of E orthogonal to Π and $div_\Pi E = \langle \nabla_X E, X\rangle + \langle \nabla_Y E, Y\rangle$, the partial divergence of the vector field E, i.e., the exponential rate of growth of the area in the direction Π under the flow in M of the vector field E.

There is a problem with this sectional curvature. It does not depend on the Weyl connection alone, it is also effected by the choice of the riemannian metric g in the conformal class. However the sign of sectional curvatures is well defined.

If M is 2-dimensional then $E_\perp = 0$. Moreover on a compact manifold M with a Weyl connection there is a unique metric in the conformal class, called the Gauduchon gauge, [Ga84], such that the vector field E is divergence free. We obtain that in dimension 2 the curvature of a Weyl connection with respect to the Gauduchon gauge is equal to the gaussian curvature of the Gauduchon gauge.

Let us summarize our discussion. We have a workable linearization (3.4) of the dynamical system (2.1) and a geometric tool (the sectional Weyl curvature (3.5)) to describe its properties. We are ready to draw conclusions about hyperbolic properties of the W–flows under the assumption of negative Weyl sectional curvature.

We introduce the Jacobi field

$$\xi = \frac{dq}{du} \quad \text{and} \quad \widehat{\eta} = \widehat{\nabla}_\xi w.$$

The Jacobi equations read

$$\frac{\widehat{D}\xi}{ds} = \widehat{\eta}, \quad \frac{\widehat{D}\widehat{\eta}}{ds} = -\widehat{R}(\xi, w)w, \tag{3.1}$$

where for any tangent vector fields X, Y,

$$\widehat{R}(X,Y) = \widehat{\nabla}_X\widehat{\nabla}_Y - \widehat{\nabla}_Y\widehat{\nabla}_X - \widehat{\nabla}_{[X,Y]}$$

is the curvature tensor of the Weyl connection.

Let us split the vector field $\widehat{\eta} = \widehat{\eta}_0 + \widehat{\eta}_1$ into the component $\widehat{\eta}_1$ orthogonal to w and the component $\widehat{\eta}_0$ parallel to w. The equations (3.1) can be split accordingly

$$\frac{\widehat{D}\widehat{\eta}_1}{ds} = -\widehat{R}_a(\xi, w)w, \quad \frac{\widehat{D}\widehat{\eta}_0}{ds} = -\widehat{R}_s(\xi, w)w, \tag{3.2}$$

where $\widehat{R}_a(X,Y)$ is the antisymmetric and $\widehat{R}_s(X,Y)$ the symmetric part of the Weyl curvature operator $\widehat{R}(X,Y) = \widehat{R}_a(X,Y) + \widehat{R}_s(X,Y)$ ($\widehat{R}_s$ is called the distance curvature and $\widehat{R}_a$ the direction curvature, cf. [F70]).

We are faced now with two tasks, to derive the linearization of (2.1) from the equations (3.1) and to study the curvature tensor of the Weyl connection.

Together with the family of Weyl geodesics let us consider the respective family of trajectories of the W–flow (2.1)

$$q(t,u), v(t,u) = \frac{dq}{dt}, |u| < \epsilon.$$

We define again the Jacobi field by $\xi = \frac{dq}{du}$. Letting $\eta = \nabla_\xi v = \nabla_v \xi$ we can consider (ξ, η) as coordinates in the tangent space of SM, which is described by $\langle v, \eta\rangle = 0$. Further we replace η with χ, the component of $\widehat{\nabla}_\xi v$ orthogonal to v. It can be calculated that

$$\chi = \eta + \langle E, v\rangle\xi - \langle \xi, v\rangle E$$

and hence we can use (ξ, χ) as linear coordinates in the tangent bundle of SM. Note that $\langle v, \eta\rangle = 0$ is equivalent to $\langle v, \chi\rangle = 0$ and in these new coordinates the velocity vector field of the W–flow (2.1) is simply $(v, 0)$. Now the linearization of (2.1) can be written as

$$\frac{\widehat{D}\xi}{dt} = \chi + \varphi(\xi)v, \quad \frac{\widehat{D}\chi}{dt} = -\widehat{R}_a(\xi, v)v + \varphi(v)\chi. \tag{3.3}$$

We will rewrite the equations (3.3) as ordinary linear differential equations with time dependent coefficients. To achieve that we need to choose frames in the tangent spaces

Theorem 2.4. *The isoenergetic dynamics*

$$\frac{dq}{dt} = v, \quad \frac{Dv}{dt} = -\nabla W + E - \frac{\langle E, v\rangle}{v^2} v. \tag{2.5}$$

on the energy level $\frac{1}{2}v^2 + W = h$*, reparametrized by the arc length defines a flow on* SM *which coincides with the W–flow for the vector field*

$$\widetilde{E} = \frac{-\nabla W + E}{2(h - W)}.$$

(We assumed implicitly that v^2 *does not vanish on the energy level set.) In particular in the isokinetic case on the energy level* $v^2 = 1$ *we obtain that the equations* (2.1) *define the W–flow for the field* E *itself.*

This theorem can be interpreted as a generalization of the Maupertuis metric,[A89]. Indeed, in the case of $E = 0$ we have

$$\widetilde{E} = \frac{-\nabla W}{2(h - W)} = -\nabla\left(-\frac{1}{2}\ln(h - W)\right),$$

and hence the Weyl connection for the field $\widetilde{E}$ is the Levi-Civita connection for the Maupertuis metric $(h - W)g$.

In this formulation it becomes transparent how the isoenergetic case differs from the isokinetic. In the former case, if $E = -\nabla U$ has a (local) potential, the vector field $\widetilde{E} = \frac{-\nabla(W+U)}{2(h-W)}$ does not have a potential unless $dW \wedge dU = 0$, i.e., unless W and U are functionally dependent. It fits well with the result of Bonetto,Cohen and Pugh [BCP99] that the isoenergetic dynamics does not in general posses the shifted hamiltonian symmetry of the Lyapunov spectrum. This result implies that the W–flows for nonpotential vector fields are not in general conformally symplectic for any choice of the conformally symplectic structure,[WL98].

§3. Jacobi equations, curvature of Weyl connections and linearizations of W–flows.

We are interested in studying hyperbolicity of the dynamical system (2.1) on SM. Since it was revealed to be a modification of the geodesic flow of a connection, it is natural to check if the Anosov theory of riemannian geodesic flows can be extended in this direction. The first step in such a study must be the investigation of linearized equations of (2.1). For riemannian geodesic flows a very useful linearization is furnished by the Jacobi equations. The Jacobi equations are valid not only for a Levi-Civita connection but for any symmetric connection. To describe them let us consider a one parameter family of geodesics of the Weyl connection, i.e., a family of solutions of (2.3) parametrized by the parameter u close to zero,

$$q(s,u), w(s,u) = \frac{dq}{ds}, |u| < \epsilon.$$

§2. Weyl manifolds and W–flows.

Let us consider a compact n-dimensional riemannian manifold M and its tangent bundle TM. The metric g will be also denoted by $\langle \cdot, \cdot \rangle$. For a smooth vector field E on M the equations of isokinetic dynamics (the Gaussian thermostat) on the energy level $v^2 = 1$ have the coordinate free form

$$\frac{dq}{dt} = v, \frac{Dv}{dt} = E - \langle E, v \rangle v, \tag{2.1}$$

where $\frac{D}{dt}$ denotes the covariant derivative, i.e., $\frac{D}{dt} = \nabla_v$ and ∇ is the Levi-Civita connection. We obtain a flow on the unit tangent bundle SM of M.

Let φ be the 1-form associated with the vector field E, i.e., $\varphi(\cdot) = \langle E, \cdot \rangle$. The form φ and the riemannian metric g define a Weyl structure on M, which is a linear symmetric connection $\widehat{\nabla}$ given by the formula (cf. [F70])

$$\widehat{\nabla}_X Y = \nabla_X Y + \varphi(Y)X + \varphi(X)Y - \langle X, Y \rangle E,$$

for any vector fields X, Y on M. The Weyl structure is usually introduced on the basis of the conformal class of g rather than g itself, but in our study we fix the riemannian metric g, which plays the role of a physical space. If we change the metric g to $\widetilde{g} = e^{-2U} g$, then the 1-form φ is replaced by $\widetilde{\varphi} = \varphi + dU$. Hence if the vector field E has a potential, i.e., $E = -\nabla U$ then the Weyl structure coincides with the Levi-Civita connection of the rescaled metric $\widetilde{g}$.

The defining property of the Weyl connection is that it is a symmetric linear connection $\widehat{\nabla}$ such that (cf. [F70])

$$\widehat{\nabla}_X g = -2\varphi(X) g, \tag{2.2}$$

for any vector field X on M, which is equivalent to the requirement that the parallel transport defined by the linear connection preserves angles. We consider the geodesics of the Weyl connection. They are given by the equations in TM

$$\frac{dq}{ds} = w, \quad \frac{\widehat{D}w}{ds} = 0, \tag{2.3}$$

where $\frac{\widehat{D}}{ds} = \widehat{\nabla}_w$. These equations provide geodesics with a distinguished parameter s defined uniquely up to scale. It follows from (2.2) and (2.3) that $\frac{d|w|}{ds} = -\varphi(w)|w|$. Assuming that at the initial point $q(0)$ we have $|w| = 1$ we obtain

$$|w| = e^{-\int_{q(0)}^{q(s)} \varphi}.$$

This formula shows that unless the form φ is exact we should not expect the geodesic flow in TM of a Weyl connection to preserve any sphere bundle. We introduce the flow

$$\Phi^t : SM \to SM,$$

which we call the *W–flow* for the field E, by parametrizing the geodesics of the Weyl connection with the arc length given by g. In other words the projection of a trajectory of Φ^t to M is a geodesic of the Weyl connection, t is the arc length parameter defined by the metric g and the trajectory itself is the natural lift of the geodesic to SM. By direct calculation we obtain

Example 1.2.

Let $\mathbb{T}^2$ be the flat torus with coordinates $(x, y) \in \mathbb{R}^2$ and $E = (a, 0)$ be the constant vector field on $\mathbb{T}^2$. The Gaussian thermostat equations on the energy level $\dot{x}^2 + \dot{y}^2 = 1$,

$$\ddot{x} = a\dot{y}^2, \quad \ddot{y} = -a\dot{x}\dot{y},$$

can be integrated and we obtain as trajectories translations of the curve

$$ax = -\ln \cos ay$$

or the horizontal lines. Assuming that E has an irrational direction on $\mathbb{T}^2$ we obtain the following global phase portrait for the isokinetic dynamics. In the unit tangent bundle $S\mathbb{T}^2 = \mathbb{T}^3$ we have two invariant tori A and R with minimal quasiperiodic motions, A contains the unit vectors in the direction of E and it is a global attractor and R contains the unit vectors opposite to E and it is a global repellor. It can be established ([W00]) that these invariant submanifolds are normally hyperbolic so that the phase portrait is preserved under perturbations. This example reveals that Gaussian thermostats, even in the most restricted isokinetic case are not in general hamiltonian with respect to any symplectic structure. In part of the phase space they may contract phase volume and hence have no absolutely continuous invariant measure.

The involution $I(q, v) = (q, -v)$ conjugates the forward and backward in time dynamics, i.e., the system is reversible. Reversibility is close to the hamiltonian property, for instance, when accompanied by enough recurrence it can replace symplecticity in KAM theory,[Se98].

Dettmann and Morriss, [DM96], proved the shifted symmetry of the Lyapunov spectrum, in the case of isokinetic dynamics with a locally potential field E by exposing the locally hamiltonian nature of the equations. For the system (1.1) with $W = 0$ and $E = -\frac{\partial U}{\partial q}$ the change of variables

$$p = e^{-U} v, \quad \frac{dt}{d\tau} = e^U,$$

brings (1.1) to the form

$$\frac{dq}{d\tau} = \frac{\partial H}{\partial p}, \quad \frac{dp}{d\tau} = -\frac{\partial H}{\partial q}, \quad H = \frac{1}{2} e^{2U} p^2 = \frac{1}{2} v^2.$$

Under the same assumptions, Choquard, [Ch97], found a variational principle, also involving the factor e^U which in the physically interesting examples is multivalued, thus making the whole description only local. Liverani and Wojtkowski, [WL98], made the observation that although the form

$$\sum dp \wedge dq = e^{-U} \left(\sum dv \wedge dq - dU \wedge (\sum v dq) \right),$$

like the coordinate system (p, q) is defined only locally, the globally defined form $\omega = \sum dv \wedge dq - dU \wedge (\sum v dq)$ can be used to develop hamiltonian-like formalism.

The three formulations above ([DM96],[Ch97],[WL98]) apply only to isokinetic dynamics with a locally potential field $E = -\nabla U$. In [W00] a geometric setup was proposed that covers all cases, i.e., isoenergetic and isokinetic, with a potential vector field E and nonpotential as well. We will describe now this setup in detail.

WEYL MANIFOLDS AND GAUSSIAN THERMOSTATS

Maciej P. Wojtkowski

University of Arizona

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Abstract. A relation between Weyl connections and Gaussian thermostats is exposed and exploited.

To Véronique

§1. Introduction.

We consider a class of mechanical dynamical systems with forcing and a thermostating term based on the Gauss' Least Constraint Principle for nonholonomic constraints, [G99]. It was originally proposed as a model for systems out of equilibrium, [HHP87],[GR97],[G99'],[R99].

Let us consider a mechanical system of the form

$$\dot{q} = v, \qquad \dot{v} = -\frac{\partial W}{\partial q} + E,$$

where q, v from $\mathbb{R}^n$ are the configuration and velocity coordinates, $W = W(q)$ is a potential function describing the interactions in the system and $E = E(q)$ is an external field acting on the system. The total energy $H = \frac{1}{2}v^2 + W$ does change because of the effect of the field E. We modify the system by applying the Gauss' Principle to the constraint $H = h$ to obtain the *isoenergetic* dynamical system

$$\dot{q} = v, \qquad \dot{v} = -\frac{\partial W}{\partial q} + E - \frac{\langle E, v\rangle}{v^2}v, \tag{1.1}$$

with possible singularities where $v = 0$. In the special case when $W = 0$ we have the *isokinetic* dynamics. The thermostating term in the equations (1.1) is called the *Gaussian thermostat*. The numerical discovery, [ECM90] that the Lyapunov spectrum, at least in the isokinetic case, has the shifted hamiltonian symmetry raised the issue of the mathematical nature of the equations (1.1).

On every energy level $H = h$ the equations (1.1) define a dynamical system. In the isokinetic case the change in h is equivalent to the appropriate rescaling of time and the multiplication of the external field E by a scalar.

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