

Revenue Management of Callable Products*

Guillermo Gallego, S. G. Kou, Robert Phillips

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Abstract

We introduce *callable products* into a finite-capacity, two-period sales or booking process where low-fare customers book first. A callable product is a unit of capacity sold at the low fare to self-selected buyers who willingly grant the capacity provider the option to “call” the capacity at a pre-specified recall price. Callable products provide a riskless source of additional revenue to the capacity provider. We calculate an optimal recall price and discount-fare booking limit for the two-period case and illustrate through a numerical study how the benefits from offering callable products can be significant, especially when high-fare demand uncertainty is large. Extensions to multi-fare structures, network models, overbooking and to other industries are discussed.

Key Words: Revenue Management, Product Design, Overbooking, Capacity Allocation

1 Introduction

The classic revenue management model studies a finite capacity, two-period sales or booking process that culminates with the delivery of a good or a service. This model divides the booking process into two periods with low-fare customers booking before high-fare customers and both fares exogenously specified. This model was motivated by the post-deregulation passenger airline industry in which price-sensitive leisure passengers book early while less price-sensitive business passengers book later after their travel uncertainty resolves. A major function of airline revenue management systems is to calculate and apply booking limits on early-booking low-fare customers, to maximize revenue by reserving sufficient space for later-booking high-fare customers. Using newsvendor-like logic to calculate profit-maximizing booking limits has been credited with generating hundreds of millions of dollars in additional revenue for airlines and industries with similar characteristics such as hotels and rental cars [19, 20].

* 331 Mudd Building, Department of Industrial Engineering and Operations Research, Columbia University, New York, NY 10027

Setting a limit on low-fare bookings is, however, an imperfect mechanism for hedging against uncertainty in future high-fare demand. Despite heavy investment in sophisticated revenue management systems, airlines lose millions of dollars a year in potential revenue; both when low-fare bookings displace higher than expected high-fare bookings (“cannibalization”) and when airlines fly empty seats protected for high-fare bookings that do not materialize (“spoilage”). Airlines could avoid cannibalization and spoilage if they could forecast future full-fare demand with certainty or, failing that, could convince business passengers to book earlier and leisure passengers to book later. Since neither of these is likely to occur, airlines are motivated to find other ways to better hedge against full-fare demand uncertainty.

In this paper we analyze the potential of low-fare “callable products” as a way for an airline to increase revenue when full-fare demand is uncertain. The concept behind a callable product is quite simple. An airline would offer both callable and standard (non-callable) products at the same low-fare during the first period. A customer purchasing the callable product would grant the airline an option to recall his seat at some future time before departure at a pre-specified recall price. Customers whose product is called would be notified by the airline sometime before departure that their seat had been called and the airline would pay the recall price. The recall price will be above the low-fare purchase price but below the full fare. The airline would call the option only if it finds that full-fare demand exceeds available capacity – that is, the remaining seats after low-fare bookings. In that case, the airline pays the recall price for a seat that it sells at the (higher) full fare.

Callable products are appealing because actions are voluntary on both sides: customers individually determine whether they wish to purchase the standard product or the callable product. The airline determines how many (if any) of the callable products it wishes to call. We show that offering callable products can generate riskless additional revenue. In §2, we describe the concept of callable products and compare the callable product approach with other mechanisms in the context of a two-period market in which low-fare customers book before high-fare customers. The effect of introducing callable products in the basic two-period revenue management model is analyzed in §3. Under very mild conditions offering callable products can generate a riskless increase in revenue –

that is, they never reduce revenue and can increase it with positive probability. Biyalogorsky et. al. [5] independently observed a similar result in a simpler model. We present first-order conditions for a globally optimal low-fare booking limit and recall price and show how these can be calculated. §4 describes numerical studies that illustrate the expected revenue gains from offering callable products. In §5 we discuss extensions to multi-fare structures, network models, overbooking and to other industries.

2 Callable Products

2.1 The Concept

When an airline offers callable products, the sequence of events is as follows:

1. The airline publishes its fares for both periods, p_L and p_H , and the recall price p .
2. Based on anticipated demands, the airline chooses a booking limit for low-fare customers.
3. First-period low-fare booking requests arrive. Each low-fare booking request is offered the choice of a standard low-fare product or a callable product. Low-fare bookings are accepted until the booking limit is reached or the end of the first period, whichever comes first. Each low-fare booking pays a fare p_L whether choosing the standard or the callable product.
4. At the end of the first period, the airline no longer offers low-fare bookings. At this point it has accepted S_L total low-fare sales of which some (possibly zero) are callable. Let V_L be the number of callable bookings and $\bar{V}_L = S_L - V_L$ the number of standard (non-callable) bookings that the airline has taken.
5. During the second period, full-fare booking requests arrive. The capacity available for full-fare bookings is $c - \bar{V}_L$ where c is the total capacity. The airline accepts full-fare booking requests until this limit is reached or the end of the second period arrives, whichever comes first.
6. If the number of full-fare bookings exceeds $c - S_L$, then the airline will call some, or all, of the callable products.

7. The airline collects p_H from every high-fare customer and pays p to every customer whose option was called. Note that the net payment to a customer whose booking gets called is $p - p_L \geq 0$.

We believe that airlines could easily implement callable products via the Internet¹. Each time a low-fare booking is made on an airline's web-site, the customer would be informed of the terms and conditions of the callable product (including the recall price, p), and asked if he would like his booking to be callable. If he agrees, then the airline would notify him (e.g. 24 hours prior to departure) via e-mail whether or not his booking had been called. If the airline chooses to exercise its call, the call price p could be credited to the customer's credit card. Of course, callable products could be offered via other channels as well.

2.2 Comparison to Alternatives

Callable products are a mechanism for suppliers to hedge against uncertainty in future high-fare demand. In addition to setting limits on discount bookings, suppliers have used a number of other mechanisms to hedge against high-fare demand uncertainty:

(1) *Stand-bys*. A stand-by booking is one sold at a deep discount, that gives the customer access to capacity only on a "space-available" basis. Customers with stand-by tickets arrive at the airport and are told at the gate whether or not they will be accommodated on their flight. If they cannot be accommodated the airline books them on a future flight (possibly also on a stand-by basis).

(2) *Bumping*. If the fares for late-booking passengers are sufficiently high, an airline could pursue a bumping strategy – that is, if unexpected high-fare demand materializes the airline would overbook with the idea that it can deny boardings to low-fare bookings in order to accommodate the high-fare passengers. For a bumping strategy to make sense, the revenue gain from the full-fare passenger must outweigh the loss from bumping the low-fare booking, including all penalties and "ill-will" cost. Historically, airlines were reluctant to overbook with the conscious intent of bumping low-fare passengers to accommodate high-fare passengers². However, with the average full-fare now

¹There is a sufficient volume of transactions for this to make sense since 40% of all airline ticket sales in North America were made on-line in 2003 with 27% of all bookings taking place on airline web sites [18].

²Of course, airlines have long overbooked as a means of hedging against cancellations and no-shows [17].

equal to seven times or more the lowest discount fare on many routes [7], the bumping strategy is beginning to make more-and-more economic sense.

(3) The *replane* concept. This service is currently being offered by the company Replane, Inc. Under the replane idea, an airline that sees higher-than-anticipated full-fare demand will contact customers with discount-fare bookings (via Internet or phone, for example) on the same flight and offer them some level of compensation to take an alternative flight.

(4) *Flexible products*. With flexible products, passengers can purchase discount tickets that ensured a seat on one of a set of flights to the same destination, with the airline having the freedom to choose which flight the customer will actually be booked on. Gallego and Phillips [10] show that offering flexible products can increase revenue by both enabling better capacity utilization and inducing additional demand.

(5) *Last-minute discounts*. The price of airline tickets generally increases as departure approaches because airlines exploit the fact that later-booking customers tend to be less price-sensitive than early-booking customers. However, increasingly airlines have been using last-minute deep discounts in order to sell capacity that would otherwise go unused³.

(6) *Auctions*. An alternative mechanism to recover previously sold capacity is to hold an auction towards the end of the booking process. An auction would allow customers to learn more about their valuation before agreeing on a price to sell back their capacity. While this information may enable the capacity provider to extract more of the ex-post consumer surplus, there are a number of complications. First, it is difficult to conceive of a practical method that would allow a large majority of the low fare customers to participate unless the auction was held at the airport very shortly before departure (this would be even more difficult to implement in other industries such as hotels and car rentals). In addition, the valuations may end up higher than predicted and the capacity provider may need to pay more than anticipated. Moreover, the capacity provider would have to make full-fare overbooking decisions before holding the auction so there is no riskless profit. Finally, there are low-fare customers that are willing to forfeit their capacity if their travel plans

³For example, the company Last-Minute Travel (www.lastminutetravel.com) specializes in selling deeply discounted capacity for flights that are nearing departure. To limit cannibalization, many airlines, hotels, and rental car companies only offer last-minute discounts through disguised (“opaque”) channels such as Priceline (www.priceline.com) or Hotwire (www.hotwire.com).

change. Under an auction these customers would need to be paid.

None of these approaches is perfect. Management of stand-bys and bumped passengers adds operational complexity and can create flight delays. Bumping is unpopular with passengers. Replaning requires searching for passengers who are willing to change flights. Flexible products require customers that are more-or-less indifferent to the actual flight on which they travel. Last-minute discounts risk cannibalizing high-fares and can train customers to wait rather than book early. Auctions are difficult to implement. By and large, callable products avoid these shortcomings.

We also note that – unlike callable products – stand-by bookings, replane, and flexible products require accommodating all booked passengers. For example, a stand-by passenger must ultimately be carried to his destination on some flight. These approaches are most effective when there is a wide disparity in capacity utilization among flights serving the same market. They allow an airline to move demand from highly utilized flights to less utilized flights, thereby freeing up capacity. These approaches are much less effective when all flights are highly utilized. In this situation, we would anticipate that callable products would be more effective since they would allow the airline to free up capacity in the market to sell to high-fare customers.⁴

All of these approaches listed above have their place. There is no reason why airlines cannot use any or all of them in combinations with callable products in order to maximize revenue for a particular flight.

3 A Two-Period Model with Callable Products

We analyze callable products in the context of a two-period, two-fare model with low-fare customers booking exclusively in the first period and high-fare customers booking exclusively in the second

⁴There is one way in which callable products might be abused. If full-fare bookings are fully refundable, a speculator could purchase a callable product, then encourage a large number of his friends to purchase full-fare tickets. By generating sufficient artificial high-fare demand, he could trigger the airline to call his ticket, after which his friends could cancel their full-fare tickets and he could pocket $p - p_L$. This abuse can be easily avoided by adding callable features only to nonrefundable tickets. Furthermore, this abuse is unlikely to be a problem in practice: First of all, the transaction costs associated with purchasing and cancelling high-fare tickets may reduce the net return to speculation substantially, since the speculator would need to create very high artificial demand in order to ensure that his booking is called. Secondly, speculation would manifest itself in a higher cancellation rate for full-fare demand. Seeing this, the airlines would reduce the number of tickets called, further increasing the costs and risks of speculation. Finally, the airlines could easily adapt current abuse-detection systems (such as those that detect and cancel multiple bookings under the same name) to detect unusual or suspicious booking patterns that might indicate speculation.

period. The two fares are exogenous and bookings are firm: that is, there are no cancellations or no-shows. This is a classic revenue management model, first studied by Littlewood [14] and extended to multiple periods and fare classes by a number of others including [2, 3, 6, 22]. We present our analysis in the context of an airline although the results are general across industries with similar characteristics (see Section 5).

3.1 The Model

An airline accepts bookings for a flight with fixed capacity c during two periods. Each booking request is for a single-unit of capacity (e.g., a single seat). Bookings occur during two periods. First-period and second-period demands are integer-valued random variables denoted by D_L and D_H respectively. We do not initially assume that D_L and D_H are independent. First-period and second-period fares are denoted by p_L and p_H respectively, with $p_H > p_L > 0$.

When an airline offers callable products, low-fare customers are given the opportunity at the time of purchase to grant the capacity provider the option of recalling their booking at a known *recall price* $p \in [p_L, p_H]$. There is no additional charge (or discount) to customers for choosing this alternative. If $p > p_L$, then customers whose reservation price for a seat is between p_L and p may choose the callable product. At the end of the second period, the provider has the opportunity to meet some of the high-fare demand in excess of residual capacity by recalling callable capacity at p and selling it for p_H .

For the capacity provider, the recall price p is a decision variable. For each customer the decision whether or not to purchase a callable product is based on the recall price p and her reservation price. We initially assume that total low-fare demand (that is, callable demand plus standard low-fare demand) is independent of the value of p . In this case, the only effect of changing p is to change the allocation of total low-fare demand between standard and callable customers. We relax this assumption in Section 4.4.

The seller's decision variables are the recall price $p \in [p_L, \bar{p}_H]$ and the low-fare booking limit $a \in \{0, \dots, c\}$. We denote the number of callables sold for a given value of a by $V_L(a)$. Then, sales at the low fare will be $S_L(a) = \min(D_L, a)$ and the number of standard (i.e. non-callable) low-fare

sales will be $\bar{V}_L(a) = S_L(a) - V_L(a)$. The capacity available for sale at the high fare is $c - \bar{V}_L(a)$, the number of units sold at the high fare is $\min(D_H, c - \bar{V}_L(a))$ and the number of units called is $\min((S_L(a) + D_H - c)^+, V_L(a))$. This means that, for any choice of the decision variables a and p , expected revenue is given by $r(a, p) = E[R(a, p)]$ where

$$R(a, p) = p_L S_L(a) + p_H \min(D_H, c - \bar{V}_L(a)) - p \min((S_L(a) + D_H - c)^+, V_L(a)). \quad (1)$$

Let $R(a)$ be the revenue corresponding to the traditional strategy without the callable product. Since the number of units sold at the low and high fares are $S_L(a)$ and $\min(c - S_L(a), D_H)$, respectively, the expected revenue from the traditional revenue management strategy of setting a discount booking limit a is given by $r(a) \equiv E[R(a)]$ where

$$R(a) = p_L S_L(a) + p_H [\min(c - S_L(a), D_H)]. \quad (2)$$

Proposition 1 *For any feasible values of a and p , the revenue realized with callable products is at least as large as the corresponding revenue without callable products with probability one. More precisely, $R(a, p) = R(a) + W(a, p)$, where $W(a, p) \geq 0$ denotes the revenue gained from callable bookings and,*

$$W(a, p) = (p_H - p) \min(G(a), V_L(a)) \geq 0, \quad G(a) := (S_L(a) + D_H - c)^+. \quad (3)$$

Proof. From (1), we have

$$\begin{aligned} R(a, p) &= p_L S_L(a) + p_H (\min(c - S_L(a), D_H)) \\ &\quad - p_H (\min(c - S_L(a), D_H)) + p_H \min(D_H, c - (S_L(a) - V_L(a))) \\ &\quad - p \min((S_L(a) + D_H - c)^+, V_L(a)) \\ &= R(a) + p_H \{ \min(D_H, c - (S_L(a) - V_L(a))) - \min(c - S_L(a), D_H) \} \\ &\quad - p \min((S_L(a) + D_H - c)^+, V_L(a)) \\ &= R(a) + (p_H - p) \min((S_L(a) + D_H - c)^+, V_L(a)) \\ &= R(a) + W(a, p) \geq R(a), \end{aligned}$$

where $W(a, p) \geq 0$ follows from $\min(G(a), V_L(a)) \geq 0$ and $p_H - p \geq 0$. $\square$

On the surface, Proposition 1 may seem to be surprising, because the revenue gain by adding the callable feature is non-negative with probability one (“riskless”). However, this is a consequence of the fact that the capacity provider will only exercise his option when it is worthwhile to do so. He can always realize the same revenue as the traditional model, with a booking limit a , by simply not calling any of the options.

Proposition 1 does not require any assumptions on $V_L(a)$. However, in what follows, we assume that low-fare customers decide whether or not to purchase the callable product independently with probability $q = g(p)$. We further assume that $q = g(p)$ is a continuous increasing function of p ⁵.

Let $\bar{p} = \inf\{p \geq p_L : g(p) = 1\}$ and let $\bar{p}_H = \min(\bar{p}, p_H)$. Clearly, the capacity provider will limit his choice of p to $p \in [p_L, \bar{p}_H]$, since there is no need to use a recall price above $\bar{p}_H$. We assume that g is strictly increasing over the interval $[p_L, \bar{p}_H]$ and denote its strictly increasing, continuous inverse by $p = h(q)$. $h(q) \equiv g^{-1}(q)$ is defined for values of $q \in [q_L, \bar{q}_H]$ where $0 \leq q_L = g(p_L) < \bar{q}_H = g(\bar{p}_H) \leq 1$. Since customers make the decision to grant the call independently with the same probability q , the number of callable units, $V_L(a)$, is conditionally binomial with parameters $S_L(a)$ and q ; i.e. $V_L(a) = \text{bino}(S_L(a), q)$. In this case we can show that under very weak conditions, the probability of positive gain from offering callable products is greater than zero.

Proposition 2 *If customers make independent decisions to grant the call with equal probability $q > 0$, $p < p_H$, and*

$$P\{S_L(a) + D_H > c, \quad S_L(a) > 0\} > 0, \quad (4)$$

then

$$P\{W(a, p) > 0\} > 0. \quad (5)$$

Proof.

$$\begin{aligned} P\{W(a, p) > 0\} &\geq P\{V_L(a) > 0 \text{ and } S_L(a) + D_H > c\} \\ &\geq P\{V_L(a) > 0 \text{ and } S_L(a) + D_H > c | S_L(a) > 0\} P\{S_L(a) > 0\} \\ &\geq q P\{S_L(a) + D_H > c | S_L(a) > 0\} P\{S_L(a) > 0\} \\ &= q P\{S_L(a) + D_H > c, \quad S_L(a) > 0\} > 0. \end{aligned}$$

⁵We will use the terms increasing and decreasing in the weak sense unless stated otherwise.

□

Notice that (4) is not particularly restrictive: it specifies that the joint probability that total bookings exceed capacity and low-fare bookings is positive. Equation (5) is similar to the concept of *arbitrage* in finance, since the revenue gain $W(a, p)$ is always nonnegative and is positive with nonzero probability. This is due in part to asymmetric information since the seller, unlike his customers, is able to observe second period, full-fare demand. It is also due in part to the fact that tickets are non-transferable. Offering the callable product results in a win-win-win situation since the capacity provider increases his revenues, low-fare customers increase their utility and high-fare customers have additional available capacity⁶.

3.2 Expected Revenue Gain

Although it is certainly good to have a “riskless” gain in revenue, it is important to understand the magnitude of the gain. If the expected additional profit is small, offering callable products may not be very interesting since the cost of implementing them may be larger than the benefit. To estimate the size of the expected gain from offering callables, we compute the mean and the distribution of $W(a, p)$ under the assumption that customers make independent decisions to choose the callable product. To do this, we use the (regularized) incomplete beta function (see Ch. 6 in [1]) which is defined for $0 \leq x \leq 1$ by

$$I_x(a, b) := \frac{B_x(a, b)}{B(a, b)}, \quad B_x(a, b) = \int_0^x t^{a-1}(1-t)^{b-1} dt, \quad \text{if } a, b > 0;$$

$$I_x(a, b) \equiv 1, \quad \text{if } a = 0; \quad I_x(a, b) \equiv 0, \quad \text{if } b = 0,$$

where $B(a, b) \equiv B_1(a, b)$ is the standard beta function. In addition,

$$0 < I_x(a, b) < 1 \text{ and } I_x(a, b) \text{ is strictly increasing in } x, \text{ if } 0 < x < 1, \quad a \neq 0, \quad b \neq 0. \quad (6)$$

⁶Notice that since buyers are allowed to decide whether or not they grant the provider the callable option, their utility is at least as large as when the callable option is not available, and there is no need to reduce the low fare to induce customers to accept the callable product. This conclusion would change if there was a secondary market for tickets. Since airline tickets are not transferable it is unlikely that such a market would ever develop in that industry. However, it would be a consideration in, for example, sporting events or Broadway shows where tickets are transferable.

Some useful properties of the incomplete beta functions are

$$I_q(a, n - a + 1) = \sum_{j=a}^n \binom{n}{j} q^j (1 - q)^{n-j}, \quad 0 \leq a \leq n + 1; \quad I_x(a, b) = 1 - I_{1-x}(b, a);$$

$$\frac{d}{dq} I_q(a, b) = \frac{q^{a-1}(1-q)^{b-1}}{B(a, b)}, \quad B(a+1, b) = \frac{a}{a+b} B(a, b), \quad a, b > 0. \quad (7)$$

Lemma 1 Suppose $X = \text{bino}(n, q)$. Then for any integer $y \geq 0$

$$E[\min(X, y)] = \begin{cases} 0, & n = 0 \text{ or } y = 0 \\ nq, & y \geq n \text{ and } n \geq 1 \\ nq\{1 - I_q(y, n - y)\} + yI_q(y + 1, n - y), & 1 \leq y \leq n - 1 \text{ and } n \geq 2 \end{cases},$$

$$\frac{d}{dq} E[\min(X, y)] = \begin{cases} 0, & n = 0 \text{ or } y = 0 \\ n, & y \geq n \text{ and } n \geq 1 \\ n\{1 - I_q(y, n - y)\}, & 1 \leq y \leq n - 1 \text{ and } n \geq 2 \end{cases}.$$

In particular, $\frac{d}{dq} E[\min(X, y)]$ is a decreasing function of q . For any $y \geq 0$ and any $x \in [0, n]$,

$$P\{\min(X, y) > x\} = \begin{cases} 0, & n = 0 \text{ or } y = 0 \\ I_q(\lfloor x \rfloor + 1, n - \lfloor x \rfloor), & y \geq x \text{ and } n \geq 1 \\ 0, & y < x \text{ and } n \geq 1 \end{cases},$$

where $\lfloor x \rfloor$ is the integer part of x .

The proof of Lemma 1 is given in the Appendix. Since

$$\begin{aligned} \{0 < G(a) \leq S_L(a) - 1, S_L(a) \geq 2\} &= \{S_L(a) + D_H > c, D_H \leq c - 1, S_L(a) \geq 2\}, \\ &= \{S_L(a) + D_H > c, D_H \leq c - 1\}, \end{aligned}$$

applying Lemma 1 immediately yields the following proposition.

Proposition 3 For any feasible value of a and p , the expected revenue gain from offering callable products is

$$\begin{aligned} E[W(a, p)] &= (p_H - p)E[\min(G(a), V_L(a))] \\ &= (p_H - p)qE[S_L(a)\{1 - I_q(G(a), S_L(a) - G(a))\}; S_L(a) + D_H > c, D_H \leq c - 1] \\ &\quad + (p_H - p)E[G(a)I_q(G(a) + 1, S_L(a) - G(a)); S_L(a) + D_H > c, D_H \leq c - 1] \\ &\quad + (p_H - p)qE[S_L(a); D_H \geq c, S_L(a) \geq 1]. \end{aligned}$$

and, for any $x \geq 0$,

$$\begin{aligned}
P\{W(a, p) > x\} \\
&= P\{\min(G(a), V_L(a)) > \frac{x}{p_H - p}\} \\
&= E[I_q(\lfloor \frac{x}{p_H - p} \rfloor + 1, S_L(a) - \lfloor \frac{x}{p_H - p} \rfloor); G(a) \geq \frac{x}{p_H - p}, S_L(a) \geq 1].
\end{aligned}$$

In the remaining sections we demonstrate that by choosing the units allocated to the low-fare customers, a , and the recall price p in an optimal way, the revenue gain can be indeed substantial.

3.3 First Order Condition for a

We assume that D_L and D_H are independent from now on.

Lemma 2 For $a \in [0, c - 1]$,

$$\Delta r(a, p) \equiv r(a + 1, p) - r(a, p) = P\{D_L > a\}[p_L - \psi(a, p)], \quad (8)$$

where, with $\bar{q} = 1 - g(p)$,

$$\psi(a, p) = (p_H - p)\bar{q}P\{D_H \geq c - \text{bino}(a, \bar{q})\} + pP\{D_H \geq c - a\}. \quad (9)$$

The proof of Lemma 2 is given in the Appendix. Consider the expression $p_L - \psi(a, p)$ inside the square bracket in equation (8) and note that it is decreasing in a . Therefore, equation (8) admits at most one sign change and this must be from positive to negative. Thus, $r(a, p)$ is unimodal in a for fixed p and the *largest maximizer* of $r(a, p)$ is given by

$$a(p) \equiv \min\{a \in [0, c] : \psi(a, p) > p_L\} \quad (10)$$

where the minimization is over the set of integers, with $a(p) = c$ if the set is empty⁷. The largest optimal booking limit for the traditional revenue management problem without callable products is given by

$$a_T^* \equiv \min\{a \in [0, c] : p_H P\{D_H \geq c - a\} > p_L\}$$

⁷Notice that if k is a positive integer such that $\psi(a(p) - k, p) = p_L$ then all the elements in the set $\{a(p) - k, \dots, a(p)\}$ maximize $r(a, p)$.

with $a_T^* = c$ if the set is empty⁸. Let b be the essential supremum of D_H , i.e., the smallest integer such that $P\{D_H \geq b\} = 0$. If D_H is unbounded, then $b = \infty$.

Lemma 3 $(c - b)^+ \leq a_T^* \leq a(p)$ for all $p \in [p_L, \bar{p}_H]$.

Proof. We will first show that $a_T^* \leq a(p)$. It is enough to prove that $\psi(a, p) \leq p_H P\{D_H \geq c - a\}$. But this is equivalent to $(p_H - p)\bar{q}P\{D_H \geq c - \text{bino}(a, \bar{q})\} \leq (p_H - p)P\{D_H \geq c - a\}$, which holds because $\bar{q} = 1 - g(p) \leq 1$ and $P\{D_H \geq c - \text{bino}(a, \bar{q})\} \leq P\{D_H \geq c - a\}$. Intuitively, we do not need to protect more than b units of capacity for high fare customers and therefore at least $(c - b)^+$ units of capacity should be made available for sale at the low fare. To make this intuition more formal we observe that if $b < c$, then at $a = c - b$ we have $p_H P\{D_H \geq c - a\} = p_H P\{D_H \geq b\} = 0$. Thus, by the definition of a_T^* , we have $a_T^* \geq c - b$, completing the proof. $\square$

It would be natural to conjecture that $a(p)$ is monotone increasing in p . However, this is not necessarily true, because the function $\psi(a, p)$ is not necessarily decreasing in p . To see this, consider the case where $q_L = 0$ and notice that $\psi(a, p_L) = \psi(a, p_H) = p_H P\{D_H \geq c - a\} \geq \psi(a, p)$. As a result, the function $r(a, p)$ may not be sub-modular and therefore we cannot invoke Topkis' Monotone Optimal Selection Theorem [21] to claim that $a(p)$ is monotone⁹.

3.4 First Order Condition for q

The following assumptions will be needed for various results in this subsection.

- **Assumption 1.** $P\{c - S_L(a) < D_H \leq c - 1\} > 0$. This assumption will be satisfied in most realistic cases and it implies condition (4).
- **Assumption 2.** $h'(q) > 0$ is an increasing function of q for $q \in [0, \bar{q}_H]$.
- **Assumption 3.** $q_L = 0$. In other words, low-fare customers will not participate in the program if the recall price is $p = p_L$.

⁸This solution to the two-period revenue management problem without callables was proposed by Littlewood [14]. Bhatia and Parekh [4], and Richter [16] demonstrate the optimality of Littlewood's formula.

⁹Intuitively, when p is close to p_L , as p increases we can allow a larger number of low-fare bookings, knowing that we can recall them and get almost the full margin $p_H - p_L$ on each recalled unit. As p approaches p_H we increasingly cannibalize high fare sales. Thus, in most applications we would expect $a(p)$ to initially increase and then decrease with p , but more complicated behavior is also possible.

Note that $R(a, p) = R(a) + W(a, p)$, where $R(a)$ does not involve q . By (3), we have

$$\begin{aligned}\frac{d}{dq}r(a, p) &= \frac{d}{dq}E[W(a, p)] \\ &= -h'(q)E[\min(G(a), V_L(a))] + (p_H - p)\frac{d}{dq}E[\min(G(a), V_L(a))],\end{aligned}\quad (11)$$

yielding the first order condition,

$$h'(q)E[\min(G(a), V_L(a))] = (p_H - p)\frac{d}{dq}E[\min(G(a), V_L(a))]. \quad (12)$$

Lemma 4 *Under Assumptions 1 and 2, $E[W(a, h(q))]$ and $r(a, h(q))$ are all strictly concave in q , $q \in [0, \bar{q}_H]$. In addition, suppose that Assumptions 1, 2, and 3 hold. If $h'(q)E[\min(G(a), V_L(a))] \geq (p_H - p)\frac{d}{dq}E[\min(G(a), V_L(a))]$ at $q = \bar{q}_H$, then (12) has a unique solution within $[0, \bar{q}_H]$; otherwise, $q = \bar{q}_H$ is optimal and is defined to be the solution of (12).*

Proof. By Assumption 2, it is sufficient to show that $\frac{d}{dq}E[\min(G(a), V_L(a))]$ is positive and strictly decreasing in q , $q \in [0, \bar{q}_H]$, because then $E[\min(G(a), V_L(a))]$ must be strictly increasing in q . To do this, note that by Lemma 1

$$\begin{aligned}\frac{d}{dq}E[\min(G(a), V_L(a))] &= E[S_L(a)\{1 - I_q(G(a), S_L(a) - G(a))\}; S_L(a) + D_H > c, D_H \leq c - 1] \\ &\quad + E[S_L(a); D_H \geq c, S_L(a) \geq 1].\end{aligned}$$

On the set $\{S_L(a) + D_H > c, D_H \leq c - 1\}$ we must have $G(a) > 0$, $S_L(a) < G(a)$, and thus $0 < I_q(G(a), S_L(a) - G(a)) < 1$, and $I_q(G(a), S_L(a) - G(a))$ is strictly increasing in q , for $0 < q < 1$, by (6). Since $E[X] > 0$ for any random variable $X \geq 0$ with $P\{X > 0\} > 0$, Assumption 1 implies that $\frac{d}{dq}E[\min(G(a), V_L(a))]$ must be positive and strictly decreasing in q , $q \in [0, \bar{q}_H]$. To check for uniqueness, notice that the left side of (12) is a strictly increasing function of q , the right side is a strictly decreasing function of q , and when $q = 0$ (at the recall price p_L) the left hand side is zero (because $V_L(a) = 0$). If at $q = \bar{q}_H$ the right side of (12) is not greater than the left side, then there must be a unique root within $[0, \bar{q}_H]$; otherwise, $\frac{d}{dq}r(a, p) > 0$ for all $p \in [p_L, \bar{q}_H]$, and $\bar{q}_H$ must be optimal. $\square$

By Lemma 1, we can write the optimality equation (12) for q as

$$h'(q)\{qE[S_L(a)\{1 - I_q(G(a), S_L(a) - G(a))\}; S_L(a) + D_H > c, D_H \leq c - 1] \quad (13)$$

$$\begin{aligned}
& +E[G(a)I_q(G(a)+1, V_L(a)-G(a)); S_L(a)+D_H > c, D_H \leq c-1] \\
& +qE[S_L(a); D_H \geq c, S_L(a) \geq 1]\} \\
= & (p_H - p)\{E[S_L(a)\{1 - I_q(G(a), S_L(a) - G(a))\}; S_L(a) + D_H > c, D_H \leq c-1] \\
& +E[S_L(a); D_H \geq c, S_L(a) \geq 1]\}.
\end{aligned}$$

For programming purposes, the terms in (13) can all be computed easily as given in Appendix.

3.5 Global Optimality

It is not immediately obvious that a value of (a, p) that satisfies the first-order conditions is necessarily a global optimum for three reasons: (1) The function $r(a, q)$ is not concave. This can be seen easily as $r(a, 0)$ is not concave in a . (2) There may be more than one global optimal solution for $r(a, q)$, as can be seen from the case of $r(a, 0)$. (3) The two parameters a and q are respectively discrete and continuous. However, the following proposition shows that any solution satisfying the first-order conditions is a global maximizer.

Proposition 4 *There must be at least one global maximizer for $r(a, p)$. Furthermore, if (a^*, p^*) is the global maximizer with the largest a^* , then we must have $a_T^* \leq a^*$. In other words, the largest optimal solution has a more generous booking limit than any traditional solution. (Note that Assumptions 1, 2, 3 are not needed for this result.)*

Proof. Since the domain $a \in [0, c]$ and $p \in [p_L, \bar{p}_H]$ is compact, $r(a, p)$ must have at least one global maximum. Next, we prove $a_T^* \leq a^*$ by contradiction. Suppose $a_T^* > a^*$. Since a_T^* is optimal for the traditional revenue management without callables, we have $E[R(a_T^*)] \geq E[R(a^*)]$. Since $E[W(a, p)]$ is increasing in a , it follows that $E[W(a_T^*, p^*)] \geq E[W(a^*, p^*)]$. Therefore, we have

$$r(a_T^*, p^*) = E[R(a_T^*)] + E[W(a_T^*, p^*)] \geq E[R(a^*)] + E[W(a^*, p^*)] = r(a^*, p^*),$$

which contradicts the fact that (a^*, p^*) is the global optimizer with the largest booking limit. $\square$

Proposition 5 *Under Assumptions 1, 2, and 3, the global maximizer (a^*, p^*) with the largest a^* must satisfy the first order condition (12) and $p^* \in (p_L, \bar{p}_H]$, and $a^* = a(g(p^*))$.*

Proof. By Assumption 1 and Proposition 1, we have that $E[W(a, p)] > 0$ for $p \in (p_L, p_H)$, and, by Assumption 3, $E[W(a, p)] = 0$ at $p = p_L$ and $p = p_H$ so it follows that any optimal solution must be in the set (p_L, p_H) . Moreover, since the profit at $\bar{p}_H$ is at least as large as the profit in $(\bar{p}_H, p_H)$ it follows that any optimal solution must be in the set $(p_L, \bar{p}_H]$. If $\bar{p}_H$ is the global optimal, then the statement is clearly true. Now let (a^*, p^*) be the global maximizer with the largest a^* , $p_L < p^* < \bar{p}_H$. Then clearly $(a^*, g(p^*))$ must satisfy the first order condition in q (Equation 12); otherwise, if the derivative is nonzero, one can always decrease (resp. increase) p^* if the derivative is less than zero (resp. greater than zero) and achieve a higher value for the objective function since q^* is in the interior. The rest of the result follows easily from Lemma 2. $\square$

Suppose Assumptions 1, 2, and 3 hold. In terms of numerical computation, if c is not large then one can first find the unique $p \in (p_L, \bar{p}_H]$ satisfying (12) for every fixed $a \in [a_T^*, c]$, and then and then find the best a through exhaustive search. If c is large, then the above search may not be efficient, and one can use Proposition 4 to eliminate many infeasible solutions while doing the search.

4 Numerical Results

In this section we present simulation results that illustrate the effects of offering callable products under different assumptions. We start by introducing the models that we will use for choosing the demands D_H and D_L and for the participation function g . We then extend the analysis to the case in which offering a callable product may actually induce additional demand.

4.1 Modelling Demand

Our demand model is based on the assumption that low and high-fare customers are drawn from disjoint populations and that all customers have a reservation price for the flight being sold. A low-fare customer will seek to book if and only if her reservation price $R_L \geq p_L$ and a high fare customer will seek to book if and only if her reservation price $R_H \geq p_H$. Initially we assume that a low-fare customer's decision to seek a booking is based only on her reservation price R_L and the fare p_L , but not on the recall price p . We relax this assumption in §4.4.

We assume that low-fare and high-fare customers arrive according to Poisson processes with rates $\lambda_{L,0}$ and $\lambda_{H,0}$, respectively. Then D_L is Poisson with mean $\lambda_L := \lambda_{L,0}P\{R_L \geq p_L\}$, and D_H is Poisson with parameter $\lambda_H := \lambda_{H,0}P\{R_H \geq p_H\}$.

We consider two possibilities. In the first case (called the ‘‘Poisson Case’’), the parameters $\lambda_{L,0}$ and $\lambda_{H,0}$ are known with certainty (i.e. they are fixed constants). In the second case, the parameters $\lambda_{L,0}$ and $\lambda_{H,0}$ are random. A general result (see p. 204 in [12]) is that if the parameter λ of a Poisson random variable has a gamma distribution with density

$$f(\lambda) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \lambda^{\alpha-1} e^{-\lambda/\beta}, \quad \lambda > 0, \alpha > 0, \beta > 0,$$

then the resulting random variable is negative binomial with

$$P(X = x) = \binom{\alpha + x - 1}{\alpha - 1} \left(\frac{\beta}{\beta + 1} \right)^x \left(\frac{1}{1 + \beta} \right)^\alpha, \quad x = 0, 1, 2, \dots$$

This is the second case (called the ‘‘Negative Binomial Case’’). The negative binomial case can also arise if customers arrive according to Poisson processes, each customer may purchase more than one ticket, and the number of tickets purchased follows a logarithmic distribution (see [12]).

4.2 The Participation Function g

Each first-period (low-fare) customer decides whether or not to purchase by comparing the low fare to his reservation price. Once he has decided to purchase he then decides whether or not to grant the call option by comparing his reservation price to the recall price. Specifically, a first-period customer will purchase if $R_L \geq p_L$ and will grant the call option if $p > R_L \geq p_L$. In this case,

$$g(p) = P\{R_L < p | R_L \geq p_L\} = 1 - P\{R_L \geq p\} / P\{R_L \geq p_L\}.$$

We extend this simple consumer-choice model in Section 4.4 to the more realistic case in which potential low-fare customers use both the low fare p_L and the recall-price p to determine whether or not they will purchase. We also note that, in practice, the participation function will most likely not be derived from a structural model but estimated directly from historical data.

4.3 Example

For all the examples, we set the capacity of the plane $c = 100$. First, we consider the case when the reservation price is uniformly distributed. For R_L uniformly distributed between $[a, b]$ with $b > p_L$, we have $g(p) = 1 - \frac{b-p}{b-p_L}$ for $p_L \leq p \leq b$, resulting in $p = h(q) = b - (1-q)(b-p_L)$, for $0 \leq q \leq 1$, which is linearly increasing and hence convex in q .

As a numerical example, suppose that reservation price is uniformly distributed between \$0 and \$300 for low-fare customers and uniformly distributed between \$0 and \$600 for high-fare customers. Also suppose the low fare $p_L = \$150$. Then $h(q) = 300 - 150(1-q) = 150 + 150q$ for $0 \leq q \leq 1$, $h'(q) = 150$ and $p = h(q) = 150 + 150q$. In this case $\bar{q}_H = 1$, and the range of q is $q \in [0, 1]$.

Table 1 shows the results of simulating the booking process with and without callable products for various settings of p_H, λ_{H0} , and λ_{L0} for both the Poisson and Negative Binomial cases. The expected revenue increase from offering callable products is quite evident. The increase is especially pronounced when the difference between low- and high-fares is significant, when high-fare demand is uncertain (the Negative Binomial Case), and when expected demand exceeds capacity. In the case of very high demand the increase due to the callables dips because the optimal allocation to low-fare bookings is small.

We want to ensure that the substantial gain from callables in Table 1 is not due to the fact that p_L and p_H were arbitrarily chosen. In order to reduce this risk, we calculate new values of p_L and p_H using the heuristic in [11]:

$$\begin{aligned} \max_{p_L, p_H} \lambda_L p_L + \lambda_H p_H &= \lambda_{L0} P\{R_L \geq p_L\} p_L + \lambda_{H0} P\{R_H \geq p_H\} p_H \\ \text{subject to } \lambda_{L0} P\{R_L \geq p_L\} + \lambda_{H0} P\{R_H \geq p_H\} &\leq c. \end{aligned} \quad (14)$$

Table 2 shows the results using p_L and p_H calculated according to (14). Note that choosing p_L and p_H in this fashion significantly improves revenue for all choices of λ_{L0} and λ_{H0} . However, the expected revenue increase from offering callable products also remains high.

We now consider the case when both R_L and R_H are exponentially distributed with means 150 and 300, respectively. Then $g(p) = 1 - e^{(p_L-p)/150} = q$, and

$$p = h(q) = p_L - 150 \log(1-q), \quad h'(q) = p_L + \frac{150}{1-q}, \quad 0 \leq q \leq q_{\max},$$

p_H	λ_{H0}	λ_{L0}	Case	Optimal Booking Limit		p^*	q^*	Expected Revenue		Ex. Rev. Increase
				Base	Call.			Base	Call.	
\$200	60	120	P	64	66	\$167.94	0.120	\$16241.0	\$16310.4	0.43%
			NB	75	77	172.39	0.149	15064.0	15160.5	0.64%
	80	160	P	52	53	170.52	0.137	17210.1	17333.4	0.73%
			NB	64	67	173.25	0.155	16290.8	16465.3	1.07%
	100	200	P	39	40	172.10	0.147	17822.4	17924.1	0.57%
			NB	53	55	173.82	0.159	16793.3	16949.3	0.93%
	150	300	P	7	7	174.68	0.165	19372.2	19393.2	0.11%
			NB	24	26	174.54	0.164	18067.8	18143.3	0.42%
	60	120	P	70	74	174.54	0.164	17798.7	17865.6	0.38%
			NB	73	81	199.09	0.327	16525.2	16866.4	2.07%
	80	160	P	60	65	181.72	0.211	20243.4	20656.7	2.04%
			NB	63	71	202.82	0.352	18548.7	19321.2	4.16%
\$300	100	200	P	50	55	186.76	0.245	21655.1	22073.3	1.93%
			NB	53	62	206.84	0.379	19770.5	20551.1	3.95%
	150	300	P	25	29	202.96	0.353	25214.7	25545.7	1.31%
			NB	28	36	216.03	0.440	22875.7	23407.8	2.33%
	60	120	P	79	84	172.24	0.148	16981.8	16990.9	0.05%
			NB	79	89	207.99	0.387	16202.2	16515.2	1.93%
	80	160	P	72	78	178.70	0.191	20815.2	21263.6	2.15%
			NB	71	83	210.72	0.405	18915.8	19982.4	5.64%
	100	200	P	65	71	182.58	0.217	22452.1	23027.9	2.57%
			NB	64	76	215.88	0.439	20379.6	21730.6	6.63%
	150	300	P	48	54	193.49	0.290	26421.4	27036.2	2.33%
			NB	46	59	229.81	0.532	23872.5	25169.2	5.43%

Table 1: Comparison of low-fare booking limits and expected revenue with and without callable products. Reservation prices are uniformly distributed on $[0, 300]$ and $[0, 600]$. “P” refers to the Poisson Case and “NB” to the Negative Binomial Case. The Negative Binomial Case has $\beta = 2$ and $\beta = 10$ for the low-fare and high-fare customers and α ’s are chosen to match the mean of the Poisson case. “Base” refers to results without callable products and “Call” to results with. Total capacity $c = 100$ and low fare price $p_L = \$150$. p^* is the optimal recall price and $q^* = g(p^*)$ is the corresponding participation probability. The last column is the expected percentage revenue increase from offering callables.

		Case	Optimal Low Fare Seats		p^*	q^*	Expected Revenue		Ex. Rev. Increase
			Base	Call.			Base	Call.	
$p_L = \$150$	$\lambda_{H0} = 60, \lambda_{L0} = 120$	P	70	74	\$174.54	0.164	\$17798.7	\$17865.6	0.38%
$p_H = \$300$	$(\lambda_H = 30, \lambda_L = 60)$	NB	73	81	199.09	0.327	16525.2	16866.4	2.07%
$p_L = 180$	$\lambda_{H0} = 80, \lambda_{L0} = 160$	P	65	70	205.15	0.210	22304.0	22577.6	1.23%
$p_H = 330$	$(\lambda_H = 36, \lambda_L = 64)$	NB	70	78	227.20	0.393	20446.2	21071.2	3.06%
$p_L = 210$	$\lambda_{H0} = 100, \lambda_{L0} = 200$	P	61	67	231.54	0.249	25747.1	26066.3	1.24%
$p_H = 360$	$(\lambda_H = 40, \lambda_L = 60)$	NB	67	77	253.49	0.483	23617.3	24360.6	3.15%
$p_L = 250$	$\lambda_{H0} = 150, \lambda_{L0} = 300$	P	52	59	267.32	0.346	31026.5	31424.6	1.28%
$p_H = 400$	$(\lambda_H = 50, \lambda_L = 50)$	NB	60	74	286.89	0.738	28406.6	29373.7	3.40%

Table 2: Comparison of results with and without callable products in the case of uniformly distributed reservation prices, when p_L and p_H are chosen according to (14). The Negative Binomial Case (NB) has $\beta = 2$ and $\beta = 10$ for low-fare and high-fare customers respectively and α 's are chosen to match the mean of the Poisson Case (P). "Base" refers to results without callable products and "Call." to results with callable products. Total capacity $c = 100$.

		Case	Optimal Low Fare Seats		p^*	q^*	Expected Revenue		Ex. Rev. Increase
			Base	Call.			Base	Call.	
$p_L = \$150$	$\lambda_{H0} = 60, \lambda_{L0} = 120$	P	78	81	\$164.95	0.095	\$13243.63	\$13243.64	0.00%
$p_H = 300$	$(\lambda_H = 22, \lambda_L = 44)$	NB	81	84	185.69	0.212	13024.9	13073.2	0.37%
$p_L = 150$	$\lambda_{H0} = 80, \lambda_{L0} = 160$	P	71	75	170.35	0.127	17517.5	17564.3	0.27%
$p_H = 300$	$(\lambda_H = 29, \lambda_L = 59)$	NB	74	80	186.28	0.215	16335.1	16609.0	1.68%
$p_L = 168$	$\lambda_{H0} = 100, \lambda_{L0} = 200$	P	66	70	191.01	0.142	20958.3	21191.5	1.11%
$p_H = 318$	$(\lambda_H = 35, \lambda_L = 65)$	NB	70	76	204.40	0.215	19203.4	19671.8	2.44%
$p_L = 243$	$\lambda_{H0} = 150, \lambda_{L0} = 300$	P	62	65	266.12	0.143	28957.5	29222.7	0.92%
$p_H = 393$	$(\lambda_H = 41, \lambda_L = 59)$	NB	69	74	276.83	0.202	26590.9	27097.8	1.91%

Table 3: Comparison of results with and without callable products when reservation prices are exponentially distributed. The Negative Binomial Case has $\beta = 2$ and $\beta = 10$ for the low fare and high fare customers and α 's are chosen to match the mean of the Poisson Case. p_L and p_H were calculated according to (14) for each case. Total capacity $c = 100$.

where $q_{\max} = 1 - e^{(p_L - p_H)/150}$. Note also that $\lambda_L = \lambda_{L0}e^{-p_L/150}$ and $\lambda_H = \lambda_{H0}e^{-p_H/300}$. Table 3 shows the results for various values of λ_{H0} and λ_{L0} when p_L and p_H have been calculated according to (14). Note that expected revenue is lower in all cases than the comparable values in Table 2. This is due to the fact that the exponential distribution results in more probability being associated with lower reservation prices than the uniform distribution. However, the incremental benefits from offering callable products are still significant.

4.4 Demand Induction

Our analysis so far has assumed that low-fare customers make their decisions to book independently of the existence of the callable product and its recall price. That is, we have assumed that a low-fare customer will seek to book if and only if his reservation price R_L is greater than or equal to the low fare, p_L . However, the option to purchase the callable product may also induce new customers who would not seek to book if the callable product were not available.

To see this, consider a low-fare customer who has a subjective probability s that his unit will be called if he purchases a callable product. His expected surplus will be $R_L - p_L$ if he purchases the standard product and $s(p - p_L) + (1 - s)(R_L - p_L)$ if he purchases a callable product. This quantity is non-negative as long as $R_L \geq p_L - (p - p_L)s/(1 - s)$. Assuming that all customers are risk-neutral, demand is induced from customers with $R_L \in [p_L - (p - p_L)s/(1 - s), p_L]$ and the probability that a buyer will agree to a call given that he makes a purchase has to be adjusted accordingly. Note that the induced customers will all purchase the callable product. The model without demand induction corresponds to the pessimistic prior $s = 0$.

Demand induction has two possible impacts: it may increase the arrival rate for the low-fare demand D_L , and it may increase the participation probability.

For example, in the case of Poisson demand, with $s = 6\%$, $a = 0$, $b = \$300$ for the low fare and $a = 0$, $b = \$600$ for the high fare the expected revenue for the case $\lambda_{L0} = 200$ and $\lambda_{H0} = 100$ (with an optimal choice of p_L and p_H) is \$26,131. This number was obtained by simulation using the booking limit and the recall price that were found optimal for the case without demand induction. This represents an increment of about \$65 relative to the case without demand induction. Of the \$65 increase, \$6 is due to the increase in the participation function and the rest is due to the increase in D_L . After optimizing over a and p using Crystal Ball we obtained an estimated expected revenue equal to \$26,154 with $a = 69$ and $p = \$241.54$.

5 Discussion and Extensions

We have introduced the concept of callable products and have shown how they can generate riskless revenue improvement in the case of a simple two-period revenue management setting. There are a

number of ways that this model simplifies reality. Real world revenue management involves multiple fare classes offered through multiple channels over many time periods, often across a network. Extension to multiple time periods and multiple fares raise a number of important design issues and potential analytical complications. However, we believe that they would not alter the fundamental insight – that offering callable products to early-booking low-fare customers would increase expected revenue when there are also later booking high-fare passengers. Of course, calculation of the booking limits for all fares would need to be modified in the presence of callable products¹⁰.

We have presented our analysis of the callable products in the traditional revenue management model in which low-fare passengers book prior to high-fare passengers. We should note that this assumption simplifies analysis and provides easy comparison with standard results, such as Littlewood’s rule. However, for callable products to improve revenue, it is not critical that all low-fare demand books before high-fare demand, it is only important that *some* low-fare demand book before *some* high-fare demand.

5.1 The Timing Issue

A more serious issue involves timing. We have assumed in this analysis that the airline can observe all high-fare demand before determining the number of calls to issues. In reality, the calls would need to be issued some time before departure (say 24 hours). There is then the probability of additional high-fare demand materializing after the decision has been made on how many bookings to call. In fact, airlines experience significant walk-up demand (sometimes called “go-shows”) that does not appear until just prior to departure. With the possibility of such late-booking high-fare demand, the decision of how many calls to issue is no longer trivial. However, even in this case, offering callable products can provide additional revenue.

To see this, consider the case in which the high-fare booking period is divided into two sub-periods. Calls need to be issued at the end of the first sub-period when some, but not all, of the high-fare bookings have arrived. Specifically, let D_{H1} be demand during the first subperiod and

¹⁰It is unlikely that a closed-form solution for the general problem of setting booking limits in the presence of callable products will be possible – rather, we would expect that airlines would use modifications of existing booking limit heuristics such as EMSR-a and EMSR-b (see Talluri and VanRyzin [20] for a discussion of these heuristics).

D_{H2} be demand during the second subperiod. It is easy to see that during the first sub-period it is optimal for the airline to first sell inventory until it is exhausted and then exercise calls until they are exhausted. Let s be the number of seats and let z be the number of calls remaining at the end of the first sub-period, after observing D_{H1} but before observing D_{H2} . The problem is to decide how many, if any, of the calls to exercise *before* the beginning of the second sub-period. We can write this problem as $\pi_2(s, z) = \min_{0 \leq x \leq z} [p_H E \min(D_{H2}, s + x) - px]$. Since the expected marginal value of exercising the $(x + 1)$ st option is $p_H Pr\{D_{H2} > s + x\} - p$ it follows that $x^* = \min[z, (y^* - s)^+]$ where y^* is the smallest integer such that $p_H P\{D_{H2} > y\} < p$.

We have assumed so far that the demand in the two sub-periods is independent. In fact, we would anticipate that the high-fare demand observed in the first subperiod would be highly correlated with the high-fare demand in the second subperiod. In this case, $y^*(d_{H1})$ would be the smallest integer such that $p_H P\{D_{H2} > y | D_{H1} = d_{H1}\} < p$ and $x^*(d_{H1})$ would be modified accordingly.

The problem of setting an booking limit and an optimal recall price is now that of maximizing

$$\max_{a,p} E[R(a) + W(a, p) + \pi_2((c - S_L(a) - D_{H1})^+, (V_L(a) - (S_L(a) + D_{H1} - c)^+)^+)].$$

where now $R(a)$ and $W(a, p)$ are defined relative to the low-fare demand and the high fare demand during the *first* sub-period.

5.2 Hedging against Cancellations and No-shows

We have presented the concept of callable tickets as a hedge against high-fare demand uncertainty. However, even in industries in which late-booking demand does not pay a higher fare, there is the opportunity to use callable products to hedge against no-shows and late cancellations. Optimal calculation of total booking limits in the face of cancellations and no-shows has been estimated to increase revenue at American Airlines by 8% or more [19]. A typical approach to setting total booking levels is for an airline to estimate a denied boarding cost B and determine the level of total bookings at which the marginal expected denied boarding cost of accepting an additional booking is equal to the expected increase in fare revenue from the additional booking (see Phillips [15] for a discussion of different overbooking approaches.) In this spirit, we consider overbooking

in the following simple model. An airline offers seats at a single fare p_L . The ticket price is entirely refundable so no-shows pay no penalty. Each “show” pays the fare but the airline must pay $B > p_L$ to each denied boarding. Let the random variable $T(y(a))$ denote the number of shows given y bookings at departure when the airline has set a total booking limit a . Then, the net revenue that the airline will receive is: $R(a) = p_L T(y(a)) - B(T(y(a)) - c)^+$, where D is demand and $y(a) = \min(D, a)$ is the number of bookings at departure given the booking limit a .

Now assume that the airline has sold $V(a)$ units of callable product with recall price p with $B > p > p_L$, and the airline is able to observe shows before deciding which customers to call. In this case, the airline would call $x = \min(T(y(a)) - c)^+, V(a)$ units and revenue would be $R'(a) = p_L T(y(a)) - px - B(T(y(a)) - x - c)^+$. Since $R'(a) \geq R(a)$ for all values of $T(y(a))$, this approach would generate a riskless savings to the airline. However, it is more realistic that the airline would need to exercise the calls some time before departure. In this case, the airline would not know whether or not the passenger being called would actually show and the revenue to the airline would be $\hat{R}(a) = p_L T(y(a)) - px - BE[T(y(a)) - x - c]^+$. The airline may have the opportunity to improve revenue by calling some bookings, however the revenue is not riskless since products need to be called prior to observing actual shows. Furthermore, using the call option in this fashion raises the issue of adverse selection – customers who choose the callable products may tend to be those who would be most likely to no-show. This means that the recall price, the total booking limit, and the number of products called would need to be jointly optimized to maximize revenue.

Callables would also provide revenue benefits to the seller in the case where there is a chance that a flight (or other event) might be canceled. In this case, the seller would only need pay a cancellation penalty of p to holders of a callable product but a higher penalty of B to ordinary ticket holders.

5.3 Pure Callable Products

We have considered the case in which each low-fare customer can choose whether he wants to purchase a standard product or a callable product. We could also consider the case where the capacity provider offers a *pure* callable product – a product in which the callable option is an

intrinsic part of the product. If the provider is selling both standard and callable products at the same price, then the situation is exactly the same from the customer’s point of view as the situation that we have analyzed – each customer can choose which product he wants to purchase. However, offering pure callable products is a more general approach since the supplier could, at any point, choose to sell only callable products. In this case, a potential customer will find that he can only purchase at the low fare if he grants the callable option. This could discourage speculators from buying tickets for shows and sporting events with the anticipation of selling them later at a higher price. Using an intensity control model, Feng and Gallego [8] show that it is never optimal to close the callable product before closing the standard product (without the call option). They also investigate the problem of dynamically selecting the recall price as a function of the number of available calls.

5.4 Callable Products with Reaccommodation

Callable products can be generalized to include passenger reaccommodation when an airline offers several different routes between the same origin and destination. Currently airlines manually reschedule the flights of customers that are bumped at the airport and of customers whose travel plans are changed by the replane mechanism. The same manual mechanism could be used to reschedule passengers whose seats are called, but we envision the development of more sophisticated reservation systems that would do this automatically.

The strategy of selling callable products can be complemented by the introduction of flexible products. While callable products pay only if the call option is exercised, flexible products offer an up-front discount to customers willing to grant the provider the option of assigning them, at a later date, to a specific product from a known pre-specified, set. As an example, a customer could buy a morning flight from New York to San Francisco at a discount with the airline assigning the customer to an actual flight at some later time. This strategy enables capacity providers to sell more capacity to high-fare, late-booking, customers but requires an up front discount and tight control over the capacity allocated to flexible products. Gallego et al. [11] develop algorithms to manage flexible products in a network where demand is driven by consumer choice models.

In a competitive environment it is also possible, maybe even desirable, to combine the features of flexible and callable products. One possibility would be to give a small up-front discount at the time of sale, as in flexibles, and an additional discount, as in callables, if the option of reallocating the customer is exercised. Another option is to sell a flexible product as a callable product that can be recalled at a pre-agreed price. Gallego et al. [11] provide a network fluid model of callable products that include elements of flexible products.

5.5 Applications in Other Industries

Although we have focused on airlines as the primary example, the analysis is applicable to any company selling constrained and perishable capacity in which some customers booking later tend to pay more than some of those booking earlier. Such companies include any of those in the traditional revenue management industries such as hotels, rental cars, cruise lines, freight carriers, and tour operators. We believe that callable products are also potentially applicable for tickets to sporting events, concerts, or shows. In these cases, the list price tends to be constant for similar quality tickets over a wide range of events (e.g. baseball games during a season) although demand can be highly variable. This has resulted in extensive secondary sales via scalpers and on-line markets such as ebay.com and stubhub.com at which tickets for popular sold-out events are traded at prices well above list. Offering callable tickets would allow a seller to maintain the same initial list-price but, if an event proves to have higher demand than expected, the seller could raise the price for later bookings, calling some earlier tickets if necessary to provide capacity.

Finally, the examples that we have discussed so far have involved consumer markets. However, we believe that there are also business-to-business selling situations in which call options might be effective. An airline, for example, may sell inventory to consolidators or re-sellers with the right to call back part of that inventory if needed. The concept also applies to industries in which low-paying demand tends to arrive early or when there is supply uncertainty or when cancellations or no-shows are commonplace. In these cases, a call option could provide a useful addition to the various business-to-business contract types surveyed in [13]. One such potential application is the situation in which a manufacturer and its customers differ in margins and demand variability. Since

manufacturers typically have more than enough capacity to serve high-margin customers who prefer to book late, they may market their excess capacity to low-margin customers with more predictable demands by providing discounts in exchange for early commitments that result in advance demand information (see [9]). Some of the low-margin customers are likely to be willing to grant a call option on contracted capacity. Similar ideas are used in the electric-power market where so-called “interruptible” contracts provide the supplier the right to interrupt service to certain customers in cases of very high demand in return for compensation – or a lower overall price.

6 Appendix

Proof of Lemma 1.

We shall only study the case when $n \geq 2$ and $1 \leq y \leq n - 1$, as the other two cases hold automatically. In this case, when $y \geq 1$,

$$\begin{aligned}
E[\min(X, y)] &= \sum_{i=1}^y i \binom{n}{i} q^i (1-q)^{n-i} + yP\{X \geq y+1\} \\
&= nq \sum_{i=1}^y \binom{n-1}{i-1} q^{i-1} (1-q)^{n-i} + yP\{X \geq y+1\} \\
&= nq \sum_{j=0}^{y-1} \binom{n-1}{j} q^j (1-q)^{n-1-j} + yP\{X \geq y+1\} \\
&= nq\{1 - I_q(y, n-y)\} + yI_q(y+1, n-y).
\end{aligned}$$

We also have

$$\begin{aligned}
\frac{d}{dq} E[\min(X, y)] &= n\{1 - I_q(y, n-y)\} + nq\left\{-\frac{q^{y-1}(1-q)^{n-y-1}}{B(y, n-y)}\right\} + y\frac{q^y(1-q)^{n-y-1}}{B(y+1, n-y)} \\
&= n\{1 - I_q(y, n-y)\} + nq\left\{-\frac{q^{y-1}(1-q)^{n-y-1}}{B(y, n-y)}\right\} + n\frac{q^y(1-q)^{n-y-1}}{B(y, n-y)} \\
&= n\{1 - I_q(y, n-y)\},
\end{aligned}$$

via (7). Finally, when $y \geq x$, $x \in [0, n]$, and $n \geq 1$, we have

$$P\{\min(X, y) > x\} = P\{X > x\} = \sum_{j=\lfloor x \rfloor + 1}^n \binom{n}{j} q^j (1-q)^{n-j} = I_q(\lfloor x \rfloor + 1, n - \lfloor x \rfloor),$$

which completes the proof.

Proof of Lemma 2.

On the set $\{D_L \leq a\}$, $S_L(a+1) = S_L(a) = D_L$, which means that $V_L(a+1)$ and $V_L(a)$ have the same distribution. Thus, $E[R(a+1) - R(a)|D_L \leq a] = 0$, from which we have

$$\Delta r(a, p) = E[R(a+1, p) - R(a, p)|D_L \geq a+1]P(D_L > a). \quad (15)$$

On the set $\{D_L \geq a+1\}$, $S_L(a+1) = a+1$ and $S_L(a) = a$. Since $E(X - Y)$ only depends on the marginal distribution of X and Y , and does not depend on the dependent structure of X and Y , we have

$$\begin{aligned} & E[\{\min(D_H, c - \bar{V}_L(a+1)) - \min(D_H, c - \bar{V}_L(a))\}|D_L \geq a+1] \\ &= E[\{\min(D_H, c - \sum_{i=1}^{a+1} \bar{\xi}_i) - \min(D_H, c - \sum_{i=1}^a \bar{\xi}_i)\}|D_L \geq a+1] \\ &= E[\min(D_H, c - \sum_{i=1}^{a+1} \bar{\xi}_i) - \min(D_H, c - \sum_{i=1}^a \bar{\xi}_i)], \end{aligned}$$

via the independence of D_L and D_H , where $\bar{\xi}_i$, $i \geq 1$, are independent Bernoulli random variables with a success probability $\bar{q}$. Since D_H is also an integer valued random variable and the values of $c - \sum_{i=1}^{a+1} \bar{\xi}_i$ and $c - \sum_{i=1}^a \bar{\xi}_i$ can only differ at most 1, we have

$$\begin{aligned} \min(D_H, c - \sum_{i=1}^{a+1} \bar{\xi}_i) - \min(D_H, c - \sum_{i=1}^a \bar{\xi}_i) &= \{c - \sum_{i=1}^{a+1} \bar{\xi}_i - (c - \sum_{i=1}^a \bar{\xi}_i)\} 1_{\{D_H \geq c - \sum_{i=1}^a \bar{\xi}_i\}} \\ &= -\bar{\xi}_{a+1} 1_{\{D_H \geq c - \sum_{i=1}^a \bar{\xi}_i\}}. \end{aligned}$$

Therefore,

$$E[\{\min(D_H, c - \bar{V}_L(a+1)) - \min(D_H, c - \bar{V}_L(a))\}|D_L \geq a+1] = -\bar{q}P\left\{D_H \geq c - \sum_{i=1}^a \bar{\xi}_i\right\}. \quad (16)$$

Similarly,

$$\begin{aligned} & E[\min((S_L(a+1) + D_H - c)^+, V_L(a+1)) - \min((S_L(a) + D_H - c)^+, V_L(a))|D_L \geq a+1] \\ &= E[\{\min((a+1 + D_H - c)^+, \sum_{i=1}^{a+1} \xi_i) - \min((a + D_H - c)^+, \sum_{i=1}^a \xi_i)\}] \\ &= P\left\{\xi_{a+1} = 0, (a+1 + D_H - c)^+ > (a + D_H - c)^+, \sum_{i=1}^a \xi_i > (a + D_H - c)^+\right\} \\ &\quad + P\left\{\xi_{a+1} = 1, (a+1 + D_H - c)^+ = (a + D_H - c)^+, (a+1 + D_H - c)^+ \geq \sum_{i=1}^{a+1} \xi_i\right\} \end{aligned}$$

$$\begin{aligned}
& +P\{\xi_{a+1}=1, (a+1+D_H-c)^+ = (a+D_H-c)^+ + 1\} \\
= & \bar{q}P\left\{\sum_{i=1}^a \xi_i > a+D_H-c \geq 0\right\} + qP\{D_H \geq c-a\} \\
= & \bar{q}\left\{P\left\{\sum_{i=1}^a \xi_i > a+D_H-c \geq 0\right\} - P\{a+D_H-c \geq 0\}\right\} + P\{D_H \geq c-a\} \\
= & -\bar{q}P\left\{\sum_{i=1}^a \xi_i \leq a+D_H-c\right\} + P\{D_H \geq c-a\}.
\end{aligned}$$

In other words,

$$\begin{aligned}
& E[\min((S_L(a+1)+D_H-c)^+, V_L(a+1)) - \min((S_L(a)+D_H-c)^+, V_L(a)) | D_L \geq a+1] \\
= & -\bar{q}P\{c - \text{bino}(a, \bar{q}) \leq D_H\} + P\{D_H \geq c-a\}.
\end{aligned} \tag{17}$$

Combining (15), (16), and (17) together yields

$$\begin{aligned}
& r(a+1, p) - r(a, p) \\
= & p_L - \bar{q}p_HP\{D_H \geq c - \text{bino}(a, \bar{q})\} + p\bar{q}P\{c - \text{bino}(a, \bar{q}) \leq D_H\} - pP\{D_H \geq c-a\},
\end{aligned}$$

which completes the proof.

Rewriting the Terms in (13) for Programming Purposes.

$$\begin{aligned}
& E[S_L(a)I_{\bar{q}}(S_L(a) - G(a), G(a)); S_L(a) + D_H > c, D_H \leq c-1] \\
= & \sum_{i=2}^a \sum_{j=0}^{c-1} P\{D_L = i, D_H = j\} i I_{\bar{q}}(c-j, i+j-c) \mathbf{1}_{\{i+j \geq c+1\}} \\
& + \sum_{j=0}^{c-1} P\{D_L \geq a+1, D_H = j\} a I_{\bar{q}}(c-j, a+j-c) \mathbf{1}_{\{a+j \geq c+1\}}, \\
& E[G(a)I_q(G(a)+1, S_L(a) - G(a)); S_L(a) + D_H > c, D_H \leq c-1] \\
= & \sum_{i=2}^a \sum_{j=0}^{c-1} P\{D_L = i, D_H = j\} (i+j-c) I_q((i+j-c+1, c-j) \mathbf{1}_{\{i+j \geq c+1\}} \\
& + \sum_{j=0}^{c-1} P\{D_L \geq a+1, D_H = j\} (a+j-c) I_q(a+j-c+1, c-j) \mathbf{1}_{\{a+j \geq c+1\}}, \\
& E[S_L(a); D_H \geq c, S_L(a) \geq 1] = \sum_{i=0}^a P\{D_L = i, D_H \geq c\} i + a \sum_{i=a+1}^{\infty} P\{D_L = i, D_H \geq c\}.
\end{aligned}$$

Also, for traditional revenue management, we have

$$\begin{aligned}
r(a) &= p_L E[S_L(a)] + p_H E[\min(c - S_L(a), D_H)] \\
&= p_L \left\{ \sum_{i=0}^a P\{D_L = i\}i + aP\{D_L \geq a + 1\} \right\} \\
&\quad + p_H \left\{ \sum_{i=0}^a \sum_{j=0}^{c-i} P\{D_L = i, D_H = j\}j + \sum_{i=0}^a (c - i)P\{D_L = i, D_H \geq c - i + 1\} \right. \\
&\quad \left. + P\{D_L \geq a + 1\} \left[\sum_{j=0}^{c-a} P\{D_H = j\}j + (c - a)P\{D_H \geq c - a + 1\} \right] \right\}.
\end{aligned}$$

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