Review problems

Perturbation Theory - Asymptotics - Euler-Lagrange Equations

Method of Characteristics and Shocks

May 3, 1999

1. Consider the quasi-linear equation

$$u_t + uu_x + au = 0, \qquad x \in R,\ t > 0,\ a > 0,$$

with initial data

$$u(x,0) = f(x),\ x \in R.$$

(a) Determine a condition on f that will guarantee that no shocks will form.

(b) If a shock does form, show that the shock time t^* is greater than for the case $a = 0$.

2. Consider the initial value problem

$$\ddot{y} + (1+\epsilon)y = 0, \qquad t > 0, \qquad y(0) = 1,\ \dot{y}(0) = 0.$$

(a) Develop a two-term perturbation approximation (assuming $0 < \epsilon << 1$). How does this approximation behave for large t ?

(b) Show that the solution of the initial value problem is bounded for all $t > 0$ and for any $\epsilon > 0$.

(c) Develop a Poincaré-Lindstedt perturbation method and find the first-order correction to the frequency. Does it agree with the result of (b) ?

3. Show that the L^2 norm of a classical solution $u(x,t)$ of the nonlinear Schrödinger equation

$$i\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + |u|^2 u, \qquad x \in R,$$

with initial data $u_0(x,0) \in L^2(R)$ is conserved.

4. Find the leading order asymptotics of the integral

$$\int_{-\infty}^{\infty} \exp\left(-\frac{x^4}{\epsilon}\right) \cos(x)\ dx,$$

as $\epsilon \to 0^+$.

5. Consider the differential equation

$$\epsilon \frac{d^4u}{dt^4} - (1+t)\frac{du}{dt} = 1 \qquad 0 < t < 1,$$

with boundary conditions $\dot{u} = u = 0$ at $t = 0$ and $t = 1$.

(a) Where are the boundary layers ?

(b) What scaling should be used in the boundary layers ?

6. Consider the functional

$$F(y) = \int_1^2 (x^2 y_x^2 + 2y^2)\ dx,$$

defined on curves $y(x)$ which satisfy the boundary conditions $y(1) = 0$ and $y(2) = 1$. Find the extremal curve of $F(y)$, subject to these boundary conditions.

7. How would you find the asymptotic behavior of

$$I(x) = \int_{-\infty}^{\infty} \exp[ix(t + t^3/3)]\ dt,$$

as $x \to +\infty$?