Review problems – Spectral theory

February 25, 1999

1. Consider the operator

$$(Lu)(x) = u(x) + \lambda \int_0^1 e^y u(y)\, dy$$

on $L^2([0,1])$, with the usual inner product.

- Find the adjoint operator L^*.
- Find the nullspace eigenvector(s) of L^*.

2. For the eigenvalue problem

$$-u'' + 4\pi^2 \int_0^1 u(x)\, dx = \lambda u, \qquad u(0) = u(1), \quad u'(0) = u'(1),$$

find the eigenvalue(s) λ and all of the associated linearly independent eigenfunctions. Is $\lambda = 0$ part of the point spectrum ?

3. Find all of the eigenvalues, λ, and associated eigenfunctions that solve the integral equation

$$\int_{-\pi}^{\pi} K(x,y)\ f(y)\, dy = \lambda f(x),$$

where $K(x,y) = 1 + \sin(x+y)$, for $x, y \in [-\pi, \pi]$.

4. Let A be a real matrix with $A^T = A$. Suppose that $Ax = \lambda x$ and $Ay = \mu y$, where x, y are vectors and λ, μ scalars, $\lambda \neq \mu$. Prove that $x \cdot y = 0$.

5. Find all $\lambda \in C$ for which the integral equation

$$\varphi(x) = \lambda \int_0^{\pi} \sin(x+y)\ \varphi(y)\, dy + f(x)$$

has a solution for every $f \in L^2([0,\pi])$.

For the λ's for which the integral equation does not have a solution for an arbitrary f, find conditions on f that guarantee existence of a solution.

Explain how the above results illustrate the Fredholm alternative. Discuss, in particular, the uniqueness of the solution in the two cases.

6. Consider the integral equation

$$u(x) = x + \frac{x^3}{3} - \int_0^x t\ u(t)\ dt.$$

Starting from $u_0(x) = 0$, find the Neumann iterates $u_1(x)$, $u_2(x)$ and $u_3(x)$. Guess the solution $u(x)$ from this information.