MATH 583 B FINAL EXAM

You have 2 hours to solve five problems. Good luck !

1. Find the leading order behavior of $\int_0^\pi e^{-xt}\ t^{-1/3}\ cos(t)\ dt$, as $x \to +\infty$.

2. (a) Find y_0 which extremizes

$$J(y) = \int_0^{\pi/2} \left(\frac{1}{2}\ y'^2 + y' + y \right)\ dx,$$

subject to $y(0) = 0$ and $y(\pi/2) = 1$.

(b) Compute $J(y_0)$.

3. Find, at leading order, the uniformly valid composite expansion of the solution to

$$\epsilon \frac{d^2y}{dt^2} - \frac{dy}{dt} + y = 0,$$

with $y(0) = 1$ and $y(1) = 0$. Here, ϵ is a small parameter.

4. Solve the integral equation

$$u(x) = \int_0^1 (x\xi + 1)\ u(\xi)\ d\xi.$$

5. Consider

$$I(x) = \int_{-\infty}^{+\infty} \frac{e^{\ i\ x\ (t+t^3/3)}}{t^2 + t_0^2}\ dt,$$

where $t_0 \neq 1$ is a real parameter. We are interested in the asymptotic behavior of $I(x)$ as $x \to +\infty$.

(a) Find the paths of steepest descent and steepest ascent going through the saddle points of the exponential part of the integrand. Draw a figure showing these paths in the complex plane.

(b) Which steepest descent path (called C_1 in the following) would you use to evaluate $I(x)$?

(c) Find the dominant contribution of the integral along C_1.

(d) Where are the poles of the integrand, in the complex plane ? Mark their location on the figure drawn in (1).

(e) Assuming the contributions along paths of equation $\mathcal{R}e(z) = T$ tend towards 0 as $T \to +\infty$, what is the leading-order behavior of $I(x)$ as $x \to +\infty$, if $t_0 > 1$?

(f) What happens when $t_0 < 1$?