Math 583 B
Homework Assignment # 9

Due April 8, 1999

Regular and Singular Perturbation Theory

1. Consider the boundary value problem

$$Ly = t^2 \frac{d^2y}{dt^2} + \epsilon\, t^2 \frac{dy}{dt} + \frac{1}{4} y = 0, \qquad 1 \le t \le e;\ 0 < \epsilon << 1,$$

with boundary conditions $y(1) = 1$ and $y(e) = 0$.

   (a) Using regular perturbation theory, find the leading order behavior $y_0(t)$.

   (b) Compute an upper bound for $|Ly_0|$ on $[1, e]$ when $\epsilon = 0.001$. Is $y_0(t)$ a good approximation to the exact solution ?

2. The equation of motion for a projectile fired from the earth's surface with initial velocity v is

$$\frac{d^2h}{dt^2} = -\frac{R^2 g}{(h+R)^2} \qquad \text{with } h(0) = 0 \text{ and } \left.\frac{dh}{dt}\right|_{t=0} = v, \tag{1}$$

where h is the height measured from the earth's surface, R is the radius of the earth, and g is the gravitational constant.

   (a) What are the dimensions of g ?

(b) Show that the variables h and t can be scaled in 3 different ways:

$$(i) \quad \tilde{t} = \frac{t}{R/v} \qquad \tilde{h} = \frac{h}{R}$$

$$(ii) \quad \tilde{t} = \frac{t}{\sqrt{R/g}} \qquad \tilde{h} = \frac{h}{R}$$

$$(iii) \quad \tilde{t} = \frac{t}{v/g} \qquad \tilde{h} = \frac{h}{v^2 g^{-1}}.$$

(c) For each of these scalings, put Eqn. (1) in dimensionless form and try to identify a small parameter ϵ.

(d) Of the 3 dimensionless equations obtained in (c), which can be sensibly solved by regular perturbation theory ?

3. We want to solve the differential equation

$$\epsilon u'' - uu' + u^3 = 0, \qquad 0 \leq x \leq 1, \tag{2}$$

subject to $u(0) = 2/3$ and $u(1) = -1/2$.

(a) Find the leading order behavior of regular solutions to

$$\epsilon u'' - uu' + u^3 = 0,$$

regardless of boundary conditions.

(b) Find the leading order behavior of the composite expansions which are solutions to Eqn. (2) and have boundary layers at $x = 0$ and $x = 1$ respectively.

(c) We now look for a solution which has an interior layer at $x = x_0$.

- Find the proper form of the inner solution, at leading order. You will have to justify the sign of the integration constant, as well as the form you are using for the solution.
- By matching this solution with the outer ones, find the value of x_0.
- Write the composite expansion of the solution at leading order.