Math 583 B
Homework Assignment # 6

Due March 11, 1999

Differential Operators - Fredholm alternative

1. Find an infinite number of eigenvalues and eigenfunctions of the following differential operator

$$L = \frac{\nabla^2}{4} + 1 - r^2,$$

where r is the radius in polar coordinates, and ∇^2 is the Laplacian in two dimensions.

Use the following information:

(a) The Gauss-Laguerre functions

$$A_n^{(m)}(r) = (2r^2)^{m/2} L_n^{(m)}(2r^2)\, e^{-r^2},$$

where m and n are positive integers, are solutions to

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right) A_n^{(m)} = 4\left(-2n - m - (1-r^2) + \frac{m^2}{4r^2}\right) A_n^{(m)}.$$

Here $L_n^{(m)}(x)$ is a Laguerre polynomial.

(b) The Laplacian in polar coordinates is

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

2. Consider the differential operator L such that

$$Lu = u'' + u, \qquad u \in L^2([0,\pi]), \quad u(0) = u(\pi) = 0.$$

(a) Find the adjoint L^* of L. Is L self-adjoint ?

(b) What is the nullspace of L ?

(c) Find the general solution u_g to $u'' + u = \sin(x)$. Can u_g satisfy the boundary conditions ? How does this illustrate the Fredholm alternative ?

3. Find the eigenvalues and eigenfunctions of the operator

$$L = -\frac{d^2}{dx^2}$$

on the set of twice differentiable functions defined on $[0, 1]$, that vanish at $x = 0$ and $x = 1$.

4. Does

$$u(x) + \int_0^1 e^{x+2y} u(y)\, dy = 1$$

have a solution in $L^2([0, 1])$? Justify your answer.