Math 583 B
Homework Assignment # 5

Due February 25, 1999

Compact Operators

1. Show that a compact set is bounded. Give an example of a bounded set which is not (sequentially) compact.

2. (a) Find the eigenfunctions of the integral operator $A \cdot = \int_0^1 K(x,\xi) \cdot d\xi$, where $K(x,\xi) = \begin{cases} x\,(1-\xi) & 0 \le x < \xi \le 1 \\ \xi\,(1-x) & 0 \le \xi < x \le 1 \end{cases}$.

   (b) Find the expansion of

   $$f(x) = \begin{cases} x/2 & 0 \le x < 1/2 \\ (1-x)/2 & 1/2 < x \le 1 \end{cases},$$

   in terms of the eigenfunctions of A. Is there a solution to $Au = f$?

3. Find the adjoint kernel for an integral operator when the inner product is

   $$< u, v >= \int_a^b u(x)\, v(x)\, \omega(x)\, dx, \qquad \omega(x) > 0.$$

4. find the resolvent operator for the integral equation

   $$y(x) = f(x) + \lambda \int_0^1 x\, e^{\xi}\, y(\xi)\, d\xi, \qquad \lambda \neq 1.$$

5. (a) Show that the solution of $u(t) = 1 + \int_0^t s \ln(s/t)\, u(s)\, ds$ satisfies the differential equation

   $$u'' + \frac{u'}{t} + u = 0 \qquad u(0) = 1, u'(0) = 0.$$

   (b) Use Neumann iterates to show that $u(t) = \sum_{k=0}^{\infty} \frac{(-1)^k (t/2)^{2k}}{(k!)^2}$.

   Remark: this function is $J_0(t)$, the zeroth order Bessel function.