# MATH 583 B MIDTERM

May 5, 1998

You have 75 minutes to solve five problems. To receive full credit, you must justify your steps with a theorem or a reason. Good luck !

## 1 Part I.

Find an infinite number of eigenvalues and eigenfunctions of the following differential operator

$$L = \frac{\nabla^2}{4} + 1 - r^2,$$

where $r$ is the radius in polar coordinates, and $\nabla^2$ is the Laplacian in two dimensions.

**Use the following information:**

**(i)** The Gauss-Laguerre functions

$$A_n^{(m)}(r) = (2r^2)^{m/2} L_n^{(m)}(2r^2)\ e^{-r^2},$$

where $m$ and $n$ are positive integers, are solutions to

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right) A_n^{(m)} = 4\left(-2n - m - (1 - r^2) + \frac{m^2}{4r^2}\right) A_n^{(m)}.$$

Here $L_n^{(m)}(x)$ is a Laguerre polynomial.

**(ii)** The Laplacian in polar coordinates is

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

## 2 Part II

Solve the following integral equation

$$y(x) = \lambda \int_{-\infty}^{0} \left( e^{x} + e^{(x+\xi)/2} \right) y(\xi)\, d\xi + F(x).$$

Your answer must include conditions on $\lambda$ and $F$ for the existence of a solution, as well as the expression of the solution if it exists.

## 3 Part III

Show that the integral operator $K$, from $L^1([0,1])$ into itself, defined by

$$(Kf)(x) = \int_0^1 \frac{f(y)}{\sqrt{1+x^2+y^2}}\, dy,$$

is bounded. The $L^1$ norm is defined by

$$||u|| = \int_0^1 |u(x)|\, dx.$$

## 4 Part IV

Consider the differential operator $L$ such that

$$Lu = u'' + u, \qquad u \in L^2([0,\pi]), \quad u(0) = u(\pi) = 0.$$

1. Find the adjoint $L^*$ of $L$. Is $L$ self-adjoint ?
2. What is the nullspace of $L$ ?
3. Find the general solution $u_g$ to $u'' + u = \sin(x)$. Can $u_g$ satisfy the boundary conditions ? How does this illustrate the Fredholm alternative ?

## 5 Part V

Starting from $u_0(x) = 0$, find the Neumann iterates $u_1(x)$, $u_2(x)$, ..., $u_n(x)$ for the integral equation

$$u(x) = 1 + \int_0^x u(t)\, dt.$$

Use the form of $u_n(x)$ to find the solution $u(x)$. Check your answer by substituting the expression of $u(x)$ in the integral equation.