MATH 583 B FINAL EXAM

May 17, 1998

You have 2 hours to solve five problems. Good luck !

Problem 1

Find the leading order behavior of

$$\int_0^{\pi} e^{-xt}\ t^{-1/3}\ cos(t)\ dt,$$

as $x \to +\infty$.

Problem 2

1. Find y_0 which extremizes

$$J(y) = \int_0^{\pi/2} \left(\frac{1}{2}\ y'^2 + y' + y \right)\ dx,$$

   subject to $y(0) = 0$ and $y(\pi/2) = 1$.

2. Compute $J(y_0)$.

Problem 3

Find, at leading order, the uniformly valid composite expansion of the solution to

$$\epsilon \frac{d^2y}{dt^2} - \frac{dy}{dt} + y = 0,$$

with $y(0) = 1$ and $y(1) = 0$. Here, ϵ is a small parameter.

Problem 4

Solve the integral equation

$$u(x) = \int_0^1 (x\xi + 1)\ u(\xi)\ d\xi.$$

Problem 5

Consider

$$I(x) = \int_{-\infty}^{+\infty} \frac{e^{\ i\ x\ (t+t^3/3)}}{t^2 + t_0^2}\ dt,$$

where $t_0 \neq 1$ is a real parameter. We are interested in the asymptotic behavior of $I(x)$ as $x \to +\infty$.

1. Find the paths of steepest descent and steepest ascent going through the saddle points of the exponential part of the integrand. Draw a figure showing these paths in the complex plane.

2. Which steepest descent path (called C_1 in the following) would you use to evaluate $I(x)$?

3. Find the dominant contribution of the integral along C_1.

4. Where are the poles of the integrand, in the complex plane ? Mark their location on the figure drawn in (1).

5. Assuming the contributions along paths of equation $\mathcal{R}e(z) = T$ tend towards 0 as $T \to +\infty$, what is the leading-order behavior of $I(x)$ as $x \to +\infty$, if $t_0 > 1$?

6. What happens when $t_0 < 1$?