Drill Problems on Ordinary Differential Equations

1. What is the type (linear, separable, exact, solvable after a change of variable, Riccati's, Bernoulli's, Clairaut's) of the following first order equations? (Do not try to solve the equations).

   (a) $(1-y^3)\sinh(3x)\ y' + 7\cosh(y)\exp(x) = 0$
   Answer: separable

   (b) $(4y + 2x + 4xy + x^2 + 3y^2)\ dx + (4x + 6y)\ dy = 0$
   Answer: e^x is an integrating factor

   (c) $[x^2 \ln(x/y)]\ dx + \dfrac{1}{x}(x^3 + 3yx^2 + \dfrac{y^4}{x})\ dy = 0$
   Answer: homogeneous of degree 2

   (d) $x^3 \sinh(x)\ y' + 5\ x^2\ y - \dfrac{3}{x}\cos(2x) = \sin(x)$
   Answer: linear

   (e) $(2x^2y - 2y + 1)\ dx - x\ dy = 0$
   Answer: linear

   (f) $[\cos(x)\sin(y) - x\sin(x)\sin(y) + y^2]\ dx + [x\cos(x)\cos(y) + 2xy]\ dy = 0$
   Answer: exact

   (g) $yy' + \cosh(x)\ y^2 = \exp(x)$
   Answer: Bernoulli with $n = -1$

2. Solve equations (e) and (f) above.

   Answer: (e): $y = -\dfrac{1}{2x^2} + \dfrac{C}{x^2}e^{x^2}$

   Answer: (f): $K = x\cos(x)\sin(y) + xy^2$

3. **(a)** Show that

$$\frac{1}{3}\ln\left[(y-1)^2|y+2|\right] = \exp(-x) - x + C$$

is an implicit solution to

$$\exp(x)(1+y)\frac{dy}{dx} + (1+\exp(x))(y^2+y-2) = 0.$$

(b) Find a solution to the above differential equation which satisfies $y(0) = 3$. Does the corresponding initial value problem have a unique solution ? Explain.

Answer: $\frac{1}{3}\ln\left[(y-1)^2|y+2|\right] = e^{-x} - x + \frac{1}{3}\ln(20) - 1$

4. **(a)** Find the general solution to $y'' - 3y' + 2y = 0$.

Answer: $y = C_1e^{2x} + C_2e^x$

Solve the initial value problem consisting of this equation and the initial condition $y(0) = 0$ and $y'(0) = 1$.

Answer: $y = e^{2x} - e^x$

(b) Is there a solution to the boundary value problem $y(0) = 0$ and $y(1) = 0$? If so, what is it ?

Answer: $y = 0$

5. Solve

$$x^3y^{(3)} - 2x^2y'' + 8xy' - 8y = 7x + x^2\cos(2\ln(x)) \qquad x > 0.$$

Answer:

$$\begin{aligned} y &= C_1x + C_2x^2\cos(2\ln(x)) + C_3x^2\sin(2\ln(x)) + \frac{7x}{5}\ln(x) \\ &\quad -\frac{x^2}{10}\ln(x)\cos(2\ln(x)) + \frac{x^2}{20}\ln(x)\sin(2\ln(x)) \end{aligned}$$

6. Solve $xy'' + y' = 0$.

Answer: $y = K_1\ln(|x|) + K_2$

7. Solve
$$y'' + 6y' + 9y = \frac{1}{x}\exp(-3x) \qquad x \neq 0.$$
Answer: $y = C_1 e^{-3x} + C_2 x e^{-3x} + x\ln(|x|)e^{-3x}$

8. Solve the following system of differential equations
$$\begin{cases} \dfrac{dx}{dt} = 2x + y + 3e^t \\ \\ \dfrac{dy}{dt} = -x + 2y - e^t \end{cases} \qquad x(0) = 0;\ y(0) = 1.$$
Answer: $\begin{cases} x = C_1\cos(t)e^{2t} + C_2\sin(t)e^{2t} - 2e^t \\ y = -C_1\sin(t)e^{2t} + C_2\cos(t)e^{2t} - e^t \end{cases}$

9. Solve $\dot{X} = AX + B$ where
$$A = \begin{pmatrix} 7 & 6 \\ 2 & 6 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} -70 \\ 35 \end{pmatrix}\exp(3t).$$
Answer: $X = \begin{pmatrix} 3e^{3t} & 2e^{10t} \\ -2e^{3t} & e^{10t} \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} + \begin{pmatrix} -60t + \frac{10}{7} \\ 40t + \frac{5}{7} \end{pmatrix} e^{3t}$

10. Solve
$$\frac{d^3y}{dt^3} - 5\frac{d^2y}{dt^2} + 12\frac{dy}{dt} - 8y = \exp(3t)\cos(2t) + 7\ t\ \exp(t).$$
Answer:
$$\begin{aligned} y &= C_1 e^t + C_2 e^{2t}\cos(2t) + C_3 e^{2t}\sin(2t) - \frac{3}{68}\ e^{3t}\cos(2t) \\ &\quad + \frac{5}{68}\ e^{3t}\sin(2t) + \left(\frac{7}{10}t^2 + \frac{14}{25}t\right)e^t \end{aligned}$$

11. Solve
$$\left[5x^4y^3 + 3y\sin(x) + 3xy\cos(x)\right]\ dx + \left[3x^5y^2 + 3x\sin(x)\right]\ dy = 0,$$
with the initial condition $y(\pi) = 5$.
Answer: $x^5y^3 + 3xy\sin(x) = 125\pi^5$