Ordinary Differential Equations Review

1 Definitions

An ordinary differential equation of order n is an equation of the form

$$\frac{d^n y}{dx^n} = F\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^{n-1}y}{dx^{n-1}}\right). \tag{1}$$

A solution to the differential equation is a n-times differentiable function $y(x)$ which satisfies (1). In the following, we will use the notation $y^{(n)} = d^n y/dx^n$.

- An *initial condition* is the prescription of the values of y and of its $(n-1)$st derivatives at a point x_0:

  $$y(x_0) = y_0,\ \frac{dy}{dx}(x_0) = y_1, \ldots \frac{d^{n-1}y}{dx^{n-1}}(x_0) = y_{n-1}, \tag{2}$$

  where y_0, y_1, ... y_{n-1} are given.

- **Existence and uniqueness theorem:**
  If F, $\partial F/\partial y$, $\partial F/\partial y'$, ..., $\partial F/\partial y^{(n-1)}$ are continuous in a neighborhood $(\alpha, \beta) \times V$ of the initial condition ($\alpha < x_0 < \beta$), then there is a unique solution to the initial value problem (1), (2) for x in some interval $(x_0 - h, x_0 + h)$.

- *Boundary conditions* prescribe the values of linear combinations of y and its derivatives for two different values of x.

In the following, we list various common techniques used to solve differential equations. Initial or boundary conditions should be imposed after the general solution has been found.

2 First order differential equations

We consider the following first order differential equation

$$M(x,y)\ dx + N(x,y)\ dy = 0.$$

2.1 Separable equations

If

$$\frac{M(x,y)}{N(x,y)} = -f(x)\ g(y),$$

then the equation is **separable**, and reads

$$f(x)\ dx = \frac{1}{g(y)}\ dy,$$

which can be integrated easily.

Example: The solution to the initial value problem

$$\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}, \qquad y(0) = -1$$

is $y = 1 - \sqrt{x^3+2x^2+2x+4}$.

2.2 M and N are both homogeneous of degree n

If $M(x,y)$ and $N(x,y)$ are both homogeneous of degree n, then the change of variable $v = y/x$ (or $u = x/y$) will make the o.d.e. separable.

Example: The general solution to $\dfrac{dy}{dx} = \dfrac{y^2+2xy}{x^2}$ is $y = \dfrac{Cx^2}{1-Cx}$, where C is an arbitrary constant.

2.3 Exact equations

If $\partial M/\partial y$ and $\partial N/\partial x$ are continuous and if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

then the equation is **exact**, i.e. there exists $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = M(x, y) \qquad \text{and} \qquad \frac{\partial u}{\partial y} = N(x, y).$$

Then the equation can be written in the form

$$\frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y}\, dy = 0,$$

which gives $d[u(x, y)] = 0$, i.e. $u(x, y) = C$. The function u can be found by integrating the two equations

$$\frac{\partial u}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial u}{\partial y} = N(x, y).$$

Example: The general solution to

$$(y\cos(x) + 2xe^y) + (\sin(x) + x^2e^y - 1)y' = 0$$

is implicitely given by $y\sin(x) + x^2e^y - y = C$, where C is an arbitrary constant.

2.4 Integrating factor

An integrating factor is a function $\rho(x, y)$ such that the differential equation $\rho(x, y)\, [M(x, y)dx + N(x, y)dy]$ is exact. If

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x},$$

but if

- $\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x} = -n\dfrac{M}{y} + m\dfrac{N}{x}$, try the integrating factor $$\rho(x, y) = x^m y^n.$$
- $\dfrac{1}{N}\left[\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right]$ is a function of x only, try the integrating factor $$\rho(x, y) = f(x).$$
- $\dfrac{1}{M}\left[\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right]$ is a function of y only, try the integrating factor $$\rho(x, y) = f(y).$$

Example: The general solution to

$$3xy + y^2 + (x^2 + xy)\frac{dy}{dx} = 0$$

is implicitely given by $x^3 y + \frac{1}{2}x^2 y^2 = C$.

2.5 Linear equations

If the equation can be written in the form

$$y' + p(x)y = q(x), \text{ i.e. if } -\frac{M(x,y)}{N(x,y)} = -p(x)y + q(x),$$

then the equation is **linear**. Let $\rho(x) = \exp\left[\int p(x)\ dx\right]$, then

$$\rho'(x) = p(x)\exp\left[\int p(x)\ dx\right] = p(x)\ \rho(x),$$

and by multiplying the o.d.e. by $\rho(x)$, we get

$$y'\rho(x) + p(x)\ \rho(x)\ y = q(x)\ \rho(x)$$

i.e.

$$\frac{d}{dx}\left[\rho(x)\ y\right] = q(x)\ \rho(x),$$

i.e.

$$y\ \rho(x) = \int q(x)\ \rho(x)\ dx + C,$$

i.e.

$$y = \frac{1}{\rho(x)}\int q(x)\ \rho(x)\ dx + \frac{C}{\rho(x)},$$

i.e.

$$y = \left[\int q(x)\ \rho(x)dx + C\right]\ \exp\left[-\int p(x)\ dx\right].$$

Examples:

1. The general solution to $y' + 2y = e^{-x}$ is $y = e^{-x} + Ce^{-2x}$.
2. The solution to $y' - 2xy = x$, $y(0) = 0$ is $y = -\frac{1}{2} + \frac{1}{2}e^{x^2}$

2.6 Bernoulli's equation

If

$$-\frac{M(x,y)}{N(x,y)} = -p(x)y + q(x)y^n, \qquad n \neq 0, 1,$$

we have the following Bernoulli's equation

$$y' + p(x)y = q(x)y^n,$$

which can be brought to the form of a linear equation by the change of variable $u = y^{1-n}$.

Example: The general solution to $x^2y' + 2xy - y^3 = 0$, $x > 0$ is $y = \pm\sqrt{\dfrac{5x}{2 + Cx^5}}$.

2.7 Riccati's equation

If

$$-\frac{M(x,y)}{N(x,y)} = a_0(x) + a_1(x)y + a_2(x)y^2,$$

we have the following Riccati's equation

$$y' = a_0(x) + a_1(x)y + a_2(x)y^2.$$

To solve it,

1. find a particular solution $y = y_1(x)$ by trying simple functions such as ax^b or $a\ \exp(bx)$.

2. $u = y - y_1$ satisfies the following Bernoulli's equation (where $n = 2$)

$$\begin{aligned} u' &= a_1(x)u + a_2(x)u\ (2y_1(x) + u) \\ &= [a_1(x) + 2\ a_2(x)\ y_1(x)]\, u + a_2(x)\ u^2, \end{aligned}$$

   which can be brought to the form of a linear equation by the change of variable $w = u^{1-2} = 1/u$, i.e. in terms of the original variable, $y = y_1 + 1/w$.

Example: The general solution to $y' = 1 + x^2 - 2xy + y^2$ is $y = x + 1/(C - x)$.

2.8 Clairaut's equation

$$y = xy' + f(y')$$

has for solution the family of straight lines

$$y = cx + f(c), \qquad c = \text{ constant}$$

and may also have a singular solution in the parametric form

$$\begin{cases} x = -f'(p) \\ y = px + f(p) \end{cases}.$$

Example: The general solution to $y = y'x + \dfrac{1}{y'}$ is $y = Cx + 1/C$ and a singular solution is $y^2 = 4x$.

3 Second Order Differential Equations

3.1 Equations without y

Second order differential equations without y are of the form

$$F(y'', y', x) = 0.$$

With $w = y'$, this equation reads $F(w', w, x) = 0$, which is a first order equation. It can then be solved for w and then for y, using $w = dy/dx$.

Example: The general solution to $x^2y'' + y'^2 - 2xy' = 0$ is $y = \dfrac{1}{2}x^2 - C_1x + C_1^2 \ln(|x + C_1|) + C_2$, where C_1 and C_2 are arbitrary constants.

3.2 Equations without x

They read

$$F(y'', y', y) = 0.$$

Let $w = dy/dx$ be a function of y. Then,

$$\frac{d^2y}{dx^2} = \frac{dw}{dx} = \frac{dw}{dy}\frac{dy}{dx} = \frac{dw}{dy}w,$$

and the ode reads

$$F(w\frac{dw}{dy}, w, y) = 0,$$

which is a first order differential equation where the dependent variable is y. One can then solve this equation for w in terms of y and then solve the first order separable equation $\frac{dy}{dx} = w(y)$.

Example: The general solution to $yy'' + (y')^2 + 4 = 0$ is implicitely given by $x^2 = \left(C_2 + \frac{1}{4C_1}\sqrt{1 - 4C_1^2y^2}\right)^2$.

3.3 Linear equations

The general solution to

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = h(x)$$

reads

$$y(x) = C_1y_1(x) + C_2y_2(x) + y_p(x),$$

where

- $y_p(x)$ is a particular solution to $a_2(x)y'' + a_1(x)y' + a_0(x)y = h(x)$,
- y_1 and y_2 are two **linearly independent** solutions of the associated homogeneous equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0,$$

- C_1 and C_2 are two constants.

Therefore, the linear equation is solved in two steps:

1. Solve the homogeneous equation (i.e. find two linearly independent solutions y_1 and y_2),
2. Find a particular solution to the full equation.

3.3.1 How to solve the homogeneous equation

The general solution y to the homogeneous equation $a_2(x)y''+a_1(x)y'+a_0(x)y = 0$ is a linear combination of two of its linearly independent solutions y_1 and y_2, i.e. $y = C_1y_1 + C_2y_2$ where C_1 and C_2 are constants.

Equations with constant coefficients
We assume that the differential equation reads

$$a_2y'' + a_1y' + a_0y = 0,$$

where a_2, a_1 and a_0 are constants. The associated characteristic equation is $a_2r^2 + a_1r + a_0 = 0$. If it has

- 2 real distinct roots r_1 and r_2, then two linearly independent solutions to the homogeneous equation are

$$y_1(x) = \exp(r_1x) \qquad \text{and} \qquad y_2(x) = \exp(r_2x),$$

- 2 complex roots $\alpha \pm i\beta$, use

$$y_1(x) = \exp(\alpha x)\cos(\beta x) \qquad y_2(x) = \exp(\alpha x)\sin(\beta x),$$

- 1 real repeated root r, use

$$y_1(x) = \exp(rx) \qquad y_2(x) = x\exp(rx).$$

Examples:

- The solution to $y'' + y' - 2y = 0$, $y(0) = 0$ and $y'(0) = 3$ is $y = e^x - e^{-2x}$.
- The general solution to $y'' - 4y' + 4y = 0$ is $y = C_1e^{2x} + C_2xe^{2x}$.

Cauchy-Euler equation
If the ode reads

$$a_2x^2y'' + a_1xy' + a_0y = 0, \quad a_2, \ a_1, \text{ and } a_0 \text{ constants},$$

we look for solutions in the form $y = x^r = \exp(r\ln(x))$ (defined for $x > 0$). Then, r must satisfy

$$a_2r(r-1) + a_1r + a_0 = 0.$$

If this equation has

- 2 real distinct roots r_1 and r_2, then two linearly independent solutions are

$$y_1 = x^{r_1} \qquad y_2 = x^{r_2},$$

- 2 complex roots $\alpha \pm i\beta$, use

$$y_1(x) = x^\alpha \cos(\beta \ln(x)) \qquad y_2(x) = x^\alpha \sin(\beta \ln(x)),$$

- 1 real repeated root r, use

$$y_1(x) = x^r \qquad y_2(x) = x^r \ln(x).$$

Note: If you have to solve for $x < 0$, first make the change of variable $t = -x$.

Example: $y = C_1 x^2$ solves $x^2 y'' - 3xy' + 4y = 0$, $y(0) = 0$, $y'(0) = 0$.

Other equations

Find a particular solution y_{ph} to the homogeneous equation and apply the method of **variation of constants**: look for another solution in the form $y = v(x) y_{ph}(x)$, substitute in the homogeneous equation and solve the first order equation for v. This procedure is called *reduction of order*.

3.3.2 How to find a particular solution to the full equation

We now turn to

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = h(x)$$

and look for a particular solution y_p to this equation.

Equations with constant coefficients: Method of undetermined coefficients

If the ode has constant coefficients and if

$$h(x) = P_m(x)\exp(\alpha x)\cos(\beta x) + Q_m(x)\exp(\alpha x)\sin(\beta x),$$

where P_m and Q_m are polynomials of degree m, then try the particular solution

- $y_p = K_m(x)\exp(\alpha x)\cos(\beta x) + L_m(x)\exp(\alpha x)\sin(\beta x)$ if $\alpha \pm i\beta$ **are not** roots of the associated characteristic equation,

- $y_p = x^h[K_m(x)\exp(\alpha x)\cos(\beta x) + L_m(x)\exp(\alpha x)\sin(\beta x)]$ if $\alpha \pm i\beta$ **are** roots of the associated characteristic equation **of multiplicity h** (h can be 1 or 2),

where K_m and L_m are polynomials of degree m in both cases.

Note: It might be useful to use the principle of superposition: if $h(x) = h_1(x) + h_2(x)$, and if y_{p1} and y_{p2} are particular solutions to $a_2(x)y'' + a_1(x)y' + a_0(x)y = h_1(x)$ and $a_2(x)y'' + a_1(x)y' + a_0(x)y = h_2(x)$ respectively, then $y_p = y_{p1} + y_{p2}$ is a particular solution to $a_2(x)y'' + a_1(x)y' + a_0(x)y = h(x)$.

Examples:

- A particular solution to $y'' + y' - 2y = 4\sin(2x)$ is $y_p = -\frac{1}{5}\cos(2x) - \frac{3}{5}\sin(2x)$.
- A particular solution to $y'' - 4y' + 4y = 12xe^{2x}$ is $y_p = 2x^3e^{2x}$.

Cauchy-Euler equations

The change of variable $x = e^t$ will turn the Cauchy-Euler equation into an equation with constant coefficients, which can then be solved as described above.

Other equations

If the ode does not have constant coefficients, or if $h(x)$ is not of the form discussed above, use the method of **variation of constants**: look for $y_p = v_1(x)y_1(x) + v_2(x)y_2(x)$ where y_1 and y_2 are two linearly independent solutions to the associated homogeneous equation and solve for v_1 and v_2 after imposing

$$v_1'(x)y_1(x) + v_2'(x)y_2(x) = 0.$$

Example: The general solution to $y'' + 2y' + y = 2x^{-2}e^{-x}$, $x > 0$ is $C_1e^{-x} + C_2xe^{-x} - 2\left(1 + \ln(x)\right)e^{-x}$.

Note:

There is a convenient way of checking that two functions y_1 and y_2 are linearly independent: their Wronskian

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

must be nonzero.

4 Linear Differential Equations of Order Higher than Two

The general solution to

$$a_n(x)y^{(n)}+a_{n-1}(x)y^{(n-1)}+\ldots+a_2(x)y''+a_1(x)y'+a_0(x)y = h(x), \qquad a_n(x) \neq 0$$

reads

$$y(x) = C_n y_n(x) + C_{n-1}y_{n-1}(x) + \ldots + C_1 y_1(x) + y_p(x),$$

where

- $y_p(x)$ is a particular solution to

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_2(x)y'' + a_1(x)y' + a_0(x)y = h(x),$$

- $y_i,\ i = 1, \ldots, n$, are n **linearly independent** solutions to the associated homogeneous equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_2(x)y'' + a_1(x)y' + a_0(x)y = 0,$$

- $C_i,\ i = 1, \ldots, n$, are n constants.

Therefore, the method to solve the linear equation is the following

1. Solve the homogeneous equation (i.e. find n linearly independent solutions $y_1, \ldots, y_n$),
2. Find a particular solution to the full equation.

4.1 How to solve the homogeneous equation

The general solution y to the homogeneous equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

is a linear combination of n of its linearly independent solutions $y_n, \ldots, y_1$, i.e. $y = C_n y_n + \ldots + C_1 y_1$ where the C_i's are constants.

4.1.1 Equations with constant coefficients

Suppose the homogeneous equation reads

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_2 y'' + a_1 y' + a_0 y = 0$$

where the a_i's are real constants. The associated characteristic equation is

$$a_n r^n + \ldots + a_1 r + a_0 = 0.$$

If it has

- only simple roots r_i, use $y_j(x) = \exp(r_j x)$ for real roots, and

  $$\begin{cases} y_i(x) &= \exp(\alpha_i x)\cos(\beta_i x) \\ y_{i+1}(x) &= \exp(\alpha_i x)\sin(\beta_i x) \end{cases}$$

  for any set of complex conjugate roots $r_i = \alpha_i + i\beta_i$ and $r_{i+1} = \alpha_i - i\beta_i$.

- one or more roots of multiplicity greater than one, use the above formula for roots of multiplicity one, and

  $$\begin{aligned} y_i &= \exp(r_i x), \\ y_{i+1} &= x\exp(r_i x), \\ y_{i+2} &= x^2\exp(r_i x), \\ &\vdots \\ y_{i+h-1} &= x^{h-1}\exp(r_i x), \end{aligned}$$

  for any root r_i of multiplicity h. Again, if $r_i = \alpha_i \pm i\beta_i$, it is customary to use $x^k \exp(\alpha_i x)\cos(\beta_i x)$ and $x^k \exp(\alpha_i x)\sin(\beta_i x)$ instead of $x^k \exp(r_i x)$ (since the ode has real coefficients).

Examples:

- The general solution to $y^{(4)} - 8y'' - 9y = 6x + 12\sin(x)$ is $y = C_1\cos(x) + C_2\sin(x) + C_3 e^{3x} + C_4 e^{-3x} - \frac{2}{3}x + \frac{3}{5}x\cos(x)$.

- The solution to $y^{(4)} - 2y^{(3)} = 0$ with $y(0) = 0$, $y'(0) = 0$, $y''(0) = 0$ and $y^{(3)}(0) = 4$ is $y = \frac{1}{2}e^{2x} - x^2 - x - \frac{1}{2}$.

4.1.2 Cauchy-Euler equation

To solve

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \ldots + a_2 x^2 y'' + a_1 x y' + a_0 y = 0, \qquad x > 0$$

where the a_i's are real constants, make the change of variable $x = \exp(t)$. This gives an equation with constant coefficients for y (as a function of t), which can be solved as described above. Then, the change of variable $t = \ln(x)$ gives y in terms of x.

Note: If you have to solve for $x < 0$, make the change of variable $x = -\exp(t)$, and proceed as before (but now, $t = \ln(-x)$).

Example: The general solution to $x^3 y^{(3)} - 2xy'' + 8xy' - 8y = 0$, $x > 0$ is $C_1 x + C_2 x^2 \cos(2\ln(x)) + C_3 x^2 \sin(2\ln(x))$.

4.2 How to find a particular solution to the full equation

We now look for a particular solution to

$$a_n(x) y^{(n)} + a_{n-1}(x) y^{(n-1)} + \ldots + a_2(x) y'' + a_1(x) y' + a_0(x) y = h(x).$$

1. If the above equation has constant coefficients and if

$$h(x) = P_m(x)\exp(\alpha x)\cos(\beta x) + Q_m(x)\exp(\alpha x)\sin(\beta x),$$

where P_m and Q_m are polynomials of degree m, then try the particular solution

- $y_p = K_m(x)\exp(\alpha x)\cos(\beta x) + L_m(x)\exp(\alpha x)\sin(\beta x)$ if $\alpha \pm i\beta$ **are not** roots of the associated characteristic equation.
- $y_p = x^h[K_m(x)\exp(\alpha x)\cos(\beta x) + L_m(x)\exp(\alpha x)\sin(\beta x)]$ if $\alpha \pm i\beta$ **are** roots of the associated characteristic equation **of multiplicity h**.

K_m and L_m are polynomials of degree m in both cases.

2. If this equation does not have constant coefficients, or if $h(x)$ is not of the form discussed above, use the method of variation of constants: look for $y_p = v_n(x)y_n(x) + \ldots + v_1(x)y_1(x)$ where the y_i's are n linearly independent solutions to the associated homogeneous equation and solve for the v_i's after imposing the $n-1$ following conditions:

$$\begin{cases} v_n'(x)y_n(x) & + & \ldots & + & v_1'(x)y_1(x) & = 0 \\ v_n'(x)y_n'(x) & + & \ldots & + & v_1'(x)y_1'(x) & = 0 \\ \vdots & & \ddots & & \vdots & \\ v_n'(x)y_n^{(n-2)}(x) & + & \ldots & + & v_1'(x)y_1^{(n-2)}(x) & = 0. \end{cases}$$

Note: If the Wronskian

$$W[y_1, \ldots, y_n] = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

of $y_1, \ldots, y_n$ is nonzero, then these functions are linearly independent.

5 Systems of First Order Linear Differential Equations

We consider systems of the form

$$\begin{cases} \frac{dx_1}{dt} & = a_{11}(t)\ x_1(t) & + a_{12}(t)\ x_2(t) & + \ \ldots & + a_{1n}(t)\ x_n(t) & + b_1(t) \\ \frac{dx_2}{dt} & = a_{21}(t)\ x_1(t) & + a_{22}(t)\ x_2(t) & + \ \ldots & + a_{2n}(t)\ x_n(t) & + b_2(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{dx_n}{dt} & = a_{n1}(t)\ x_1(t) & + a_{n2}(t)\ x_2(t) & + \ \ldots & + a_{nn}(t)\ x_n(t) & + b_n(t) \end{cases},$$

which can also be written as

$$DX = \frac{d}{dt}X = \dot{X} = AX + B,$$

where

$$X = X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \qquad B = \begin{bmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{bmatrix},$$

and

$$A = A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{bmatrix}.$$

If B and the a_{ij}'s are continuous on an interval $a < t < b$, and if $t_0 \in (a, b)$, then there is a unique solution to $\dot{X} = AX + B$ which satisfies $x_1(t) = \alpha_1,\ x_2(t) = \alpha_2,\ \dots, x_n(t) = \alpha_n$, valid on the interval (a, b).

The general solution to $\dot{X} = AX + B$ reads

$$\begin{aligned} X(t) &= U(t)C + X_p(t), \\ &= C_1 X_1(t) + C_2 X_2(t) + \ldots + C_n X_n(t) + X_p(t), \end{aligned}$$

where

- $X_p(t)$ is a particular solution to $\dot{X} = AX + B$,
- $X_i,\ i = 1, \dots, n$, are n **linearly independent** solutions to the associated homogeneous system $\dot{X} = AX$,
- $C_i,\ i = 1, \dots, n$, are n constants.

The matrix $U(t) = [X_1(t), X_2(t), \dots, X_n(t)]$ is called **a fundamental matrix** associated with the homogeneous system $\dot{X} = AX$.

Therefore, to solve the differential system,

1. Solve the homogeneous system (i.e. find n linearly independent solutions $X_1,\ \dots,\ X_n$),
2. Find a particular solution to the full system.

5.1 How to solve the homogeneous system

The general solution $X(t)$ to the homogeneous system $\dot{X} = AX$ is a linear combination of n linearly independent solutions $X_1, \ldots, X_n$, i.e. $X(t) = C_1X_1(t) + \ldots + C_nX_n(t)$ where the C_i's are constants. In other words, it reads

$$X(t) = U(t)C$$

where $U(t)$ is a fundamental matrix and C a constant vector.

In the following, we consider systems with constant coefficients only, that is we assume the matrix A does not depend on time. The procedure is then as follows.

1. Find the eigenvalues and eigenvectors of A.

2. For each eigenvector ξ belonging to the eigenvalue λ, $X(t) = \exp(\lambda t)\xi$ is a solution to $\dot{X} = AX$. Therefore,

   - if A has n distinct eigenvalues, we know n linearly independent solutions to the homogeneous system, namely $X_i(t) = \exp(\lambda_i t)\xi_i$, $i = 1, \ldots, n$.

   - if an eigenvalue λ has a multiplicity h higher than one, find as many linearly independent eigenvectors as you can, and write the corresponding solutions $X_j(t) = \exp(\lambda t)\xi_j$. Then, look for the missing solutions in the form

     $$X_i = (t^{h-1}Y_{h-1} + t^{h-2}Y_{h-2} + \ldots + tY_1 + Y_0)\exp(\lambda t),$$

     where the Y_i's are constant vectors. Substitute this solution in the homogeneous system and solve for the Y_i's. Always make sure that you have found n linearly independent solutions to the homogeneous system, and write the corresponding fundamental matrix.

Note: If one eigenvalue is complex, and since (in general) A has real coefficients, its complex conjugate is also an eigenvalue. In the same way, if ξ is a (complex) eigenvector that belongs to the complex eigenvalue λ, then its complex conjugate ξ^* belongs to the eigenvalue λ^*.

So, if one eigenvalue $\lambda = \alpha + i\beta$ is complex, you may use the real functions $X_i(t) = \Re e[\exp(\alpha + i\beta)\xi]$ and $X_{i+1}(t) = \Im m[\exp(\alpha + i\beta)\xi]$ instead of the complex solutions $X_i(t) = \exp(\lambda t)\xi$ and $X_{i+1}(t) = X_i^*(t) = \exp(\lambda^* t)\xi^*$.

Example:

- A fundamental matrix for $\dot{X} = AX$ with $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$ is $\begin{bmatrix} 3e^{-t} & e^{4t} \\ -2e^{-t} & e^{4t} \end{bmatrix}$.
- A fundamental matrix for $\dot{X} = AX$ with $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$ is $\begin{bmatrix} e^{2t} & te^{2t} \\ e^{2t} & (t+1)e^{2t} \end{bmatrix}$.

5.2 How to find a particular solution to the full system

$$\dot{X} = AX + B(t).$$

1. If $B(t) = b\exp(\omega t)$, where b is a constant vector, and if ω **is not** an eigenvalue of A, then try

   $$X_p(t) = k\exp(\omega t)$$

   where k is a constant vector. Plug the expression of X_p back into the differential system and solve for k.

   Notes:

   (a) It might be useful to use the principle of superposition.

   (b) Since A has real coefficients, if X_p is a solution to $\dot{X} = AX + b\exp(\alpha t + i\beta t)$ (where b is a constant vector and is real), then $\Re e(X_p)$ and $\Im m(X_p)$ are respectively solutions to $\dot{X} = AX + b\exp(\alpha t)\cos(\beta t)$ and to $\dot{X} = AX + b\exp(\alpha t)\sin(\beta t)$.

   Thus, if $B(t)$ has the form $b\exp(\alpha t)\cos(\beta t)$ or $b\exp(\alpha t)\sin(\beta t)$, and if $\alpha + i\beta$ is not an eigenvalue of A, the above method can be used to find a particular solution to $\dot{X} = AX + b\exp(\alpha t + i\beta t)$. Its real or its imaginary part (depending on the form of $B(t)$) is then a particular solution to the original differential system.

2. If the above method cannot be applied, use the method of **variation of constants**: look for
$$X_p = U(t)V(t)$$
where $U(t)$ is a fundamental matrix associated with the homogeneous system, and $V(t)$ a vector to be determined. Then $V(t)$ satisfies
$$\dot{V} = U^{-1}(t)B(t) \qquad \text{i.e.} \qquad V(t) = \int U^{-1}(t)B(t)dt.$$

Examples:

- A particular solution to $\dot{X} = AX + H$ with $A = \begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix}$ and $H = \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^t$ is $X_p = \begin{bmatrix} 0 \\ -1/2 \end{bmatrix} e^t$.

- A particular solution to $\dot{X} = AX + H$ with $A = \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix}$ and $H = \begin{bmatrix} 2 \\ 4 \end{bmatrix} e^t$ is $X_p = \begin{bmatrix} 5t \\ -\frac{3}{2}e^{2t} \end{bmatrix}$.

General remark: Sometimes, the elimination method (which consists in making successive operations on the rows of the linear system) may be easier to apply, or may simply go faster. It may also be useful if the matrix A depends on time.

References:

- W.E. Boyce and R.C. DiPrima, *Elementary Differential Equations and Boundary Value Problems*, John Wiley & Sons, New York, 1986.

- E.L. Ince, *Ordinary Differential Equations*, Dover, New York, 1956.