Drill Problems on Linear Algebra *

1. Which of the following subsets of $\mathbf{R}^3$ are actually subspaces ?

   (a) The set of vectors b with first component $b_1 = 0$.

   (b) The set of vectors b with $b_1 = 1$.

   (c) The vectors $b = (b_1, b_2, b_3)$ with $b_1 b_2 = 0$.

   (d) The solitary vector $b = (0, 0, 0)$.

   (e) All combinations of two given vectors $x = (1, 1, 0)$ and $y = (2, 0, 1)$.

   (f) The vectors $b = (b_1, b_2, b_3)$ which satisfy $b_3 - b_2 + 3b_1 = 0$.

2. Describe the column space and the nullspace of the following matrices

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

3. Under what conditions on b_1 and b_2 (if any) does $Ax = b$ have a solution, if

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 0 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} ?$$

   Find two vectors x in the nullspace of A, and the general solution to $Ax = b$.

*from G. strang, *Linear Algebra and its Applications*, Harcourt Brace Jovanovich, Publishers, San Diego 1986

4. Find a counterexample to the following statement: if $v_1,\ v_2,\ v_3,\ v_4$ is a basis for the vector space $\mathbf{R}^4$, and if W is a subspace, then some subset of the v's is a basis for W.

5. True or false:

   (a) If the columns of A are linearly independent, then $Ax = b$ has exactly one solution for every b.

   (b) A 5 by 7 matrix never has linearly independent columns.

6. Suppose V is a vector space of dimension 7 and W is a subspace of dimension 4. True or false:

   (a) Every basis for W can be extended to a basis for V by adding three more vectors.

   (b) Every basis for V can be reduced to a basis for W by removing three vectors.

7. If A is a 64 by 17 matrix of rank 11, how many independent vectors satisfy $Ax = 0$? How many independent vectors satisfy $A^T y = 0$?

8. If A is a real $m \times n$ matrix and if $Ax = b$ always has at least one solution, show that the only solution to $A^T y = 0$ is $y = 0$. (hint: what is the rank of A ?)

9. If A is a real $m \times n$ matrix and if $Ax = 0$ has a nonzero solution, show that $A^T y = f$ fails to be solvable for some right side f. Construct an example of A and f.

10. Find the rank and nullspace of each of the following matrices

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

11. Find an orthogonal basis for the subspace spanned by $a_1 = (1, -1, 0, 0)$, $a_2 = (0, 1, -1, 0)$ and $a_3 = (0, 0, 1, -1)$.