Drill Problems on Complex Numbers *

1. Express each of the following numbers in complex exponential form
   (a) 1
   (b) $-i$
   (c) $1+i$
   (d) $\frac{1}{2}+i\frac{\sqrt{3}}{2}$
   (e) $\frac{1}{2}-i\frac{\sqrt{3}}{2}$
2. Express each of the following complex numbers in the form $x+iy$
   (a) $e^{2+i\pi/2}$
   (b) $\frac{1}{1+i}$
   (c) $(1+i)^3$
   (d) $|3+4i|$
3. Find all of the roots of $z^3=4$ and $z^4=-1$.
4. Establish the following results:
   (a) $\overline{z+w}=\bar{z}+\bar{w}$
   (b) $|z-w|\leq|z|+|w|$
   (c) $|\,|z|-|w|\,|\leq|z+w|$
   (d) $z-\bar{z}=2i\Im m(z)$

*Based on Mark J. Ablowitz and Athanassios S. Fokas, *Complex Variables*, Cambridge Texts in Applied Mathematics, Cambridge University Press, 1997

(e) $\Re e(z) \leq |z|$

(f) $|w\bar{z} + z\bar{w}| \leq 2|wz|$

(g) $|zw| = |z||w|$

5. Use the series representation

$$e^z = \sum_{j=0}^{\infty} \frac{z^j}{j!}, \qquad |z| < \infty$$

to determine series representations for $\cos(z)$ and $\sin(z)$.

6. Use the relationship $e^{i(x_1+x_2)} = e^{ix_1}e^{ix_2}$ where x_1 and x_2 are real to deduce the known trigonometric formulae

$$\sin(x_1 + x_2) = \sin(x_1)\cos(x_2) + \cos(x_1)\sin(x_2)$$
$$\cos(x_1 + x_2) = \cos(x_1)\cos(x_2) - \sin(x_1)\sin(x_2)$$

7. Use Euler's formula to express $\cos^3(x)$ in terms of $\cos(3x)$ and $\cos(x)$.