Vector Calculus
Review

1 The gradient

The gradient of a differentiable function of three variables $f(x, y, z)$ is defined by

$$\text{grad}f = \nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

and has the following properties:

- ∇f points in the direction where f is increasing the fastest.
- The rate of change of f in that direction is equal to $||\nabla f||$.
- ∇f is perpendicular to the level surfaces of f (i.e. to surfaces of equation $f = C$=constant).
- The directional derivative of f in the direction of the **unit vector** $\vec{u}$ is equal to $\nabla f \cdot \vec{u}$.
- Critical points of f are such that $\nabla f = \vec{0}$ at these points. Generic critical points are minima, maxima and saddle points of the function f.
- The extrema of f subject to the constraint $g(x, y, z) = C$ are points on the curve of equation $g(x, y, z) = C$ where the gradient of f is parallel to the gradient of g. The constant of proportionality is called a *Lagrange multiplier*.

2 Line integrals and gradient fields

The line integral of a continuous vector field $\vec{F}$ along the oriented path C is written

$$\int_C \vec{F}(\vec{r}) \cdot d\vec{r},$$

where $\vec{r}$ is the position vector and $d\vec{r}$ is tangent to the path C. If we know a parametrization of C, i.e. if C is drawn by $\vec{r}(t)$ when t varies between t_0 and t_1, the above line integral reads

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_1} \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\, dt,$$

and can thus be computed.

Examples

- $\vec{F}(x, y) = e^x\vec{i} + e^y\vec{j}$ and C is the part of the ellipse $x^2 + 4y^2 = 4$, joining $(0, 1)$ to $(2, 0)$ in the clockwise direction.

  A parametrization of C is $x = 2\sin(t)$, $y = \cos(t)$, with t varying between 0 and $\pi/2$. Therefore,

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_0^{\pi/2} \left[e^{2\sin(t)}\vec{i} + e^{\cos(t)}\vec{j}\right] \cdot \left[2\cos(t)\vec{i} - \sin(t)\vec{j}\right] dt \\
&= \int_0^{\pi/2} \left[2\cos(t)e^{2\sin(t)} - \sin(t)e^{\cos(t)}\right]\, dt \\
&= \left[e^{2\sin(t)} + e^{\cos(t)}\right]_0^{\pi/2} = e(e-1).
\end{aligned}$$

- The work done by the force $\vec{F}$ along the path C is given by $W = \int_C \vec{F} \cdot d\vec{r}$. With Newton's law, $\vec{F} = m\, d^2\vec{r}/dt^2$, so that

$$W = \int_{t_0}^{t_1} m\, \frac{d^2\vec{r}}{dt^2} \cdot \frac{d\vec{r}}{dt}\, dt = \left[\frac{1}{2} m \left(\frac{d\vec{r}}{dt}\right)^2\right]_{t_0}^{t_1} = \left[\frac{1}{2} m v^2\right]_{t_0}^{t_1},$$

  i.e. the work done by $\vec{F}$ is equal to the gain in kinetic energy between the end points.

A vector field $\vec{F}$ is a *gradient field* if there is a function $f(x, y, z)$ such that $\vec{F} = \nabla f$. The vector field $\vec{F}$ is then *conservative*, *path-independent* and *circulation free*.

- A *path-independent* vector field $\vec{F}$ is such that the line integral of $\vec{F}$ along any path in the domain of $\vec{F}$ only depends on the end points.

- A *circulation free* vector field is such that its circulation (i.e. its line integral along any closed curve) is zero everywhere in the domain of $\vec{F}$.

- The **fundamental theorem of calculus for line integrals** tells us that a gradient field is path-independent, i.e.

$$\int_C \nabla f \cdot d\vec{r} = f(Q) - f(P),$$

  where C is a (piecewise) smooth path joining P to Q and ∇f is continuous on C. Conversely, any path-independent vector field is a gradient field.

  Example:

  Consider the force $\vec{F} = m\frac{d^2\vec{r}}{dt^2}$ discussed above. If $\vec{F}$ has a potential function, i.e. if there exists a function $V(r)$ such that $\vec{F} = -\nabla V$, we have

$$\begin{aligned} W &= \int_C \vec{F} \cdot d\vec{r} = -\int_C \nabla V \cdot d\vec{r} = V(\vec{r}(t_0)) - V(\vec{r}(t_1)) \\ &= \left[\frac{1}{2}mv^2\right]_{t_0}^{t_1}. \end{aligned}$$

  Thus,

$$\frac{1}{2}mv^2(t_0) + V(\vec{r}(t_0)) = \frac{1}{2}mv^2(t_1) + V(\vec{r}(t_1)),$$

  i.e. the total energy, which is the sum of the kinetic and potential energies, is conserved.

- If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ is a gradient field with continuous partial derivatives, then

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \qquad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}, \qquad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z},$$

  i.e. $\text{curl}\vec{F} = \vec{0}$.

3 The curl

The curl of a vector field $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ with continuous partial derivatives is a vector given by

$$\begin{aligned} \text{curl}\vec{F} &= \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}. \end{aligned}$$

It has the following properties:

- The direction of $\text{curl}\vec{F}$ at a point P is the direction around which the circulation density of $\vec{F}$ is the greatest.
- The magnitude of $\text{curl}\vec{F}$ is the circulation density around that direction.
- The circulation density of $\vec{F}$ around any unit vector $\vec{n}$ is equal to $\text{curl}\vec{F} \cdot \vec{n}$.
- For any function f with continuous second partial derivatives,

  $$\text{curl grad}f = \nabla \times \nabla f = \vec{0}.$$

  Conversely, any smooth vector field defined on a domain with no holes and whose curl is zero everywhere is a gradient field (this is the *curl test*).

4 Stokes's and Green's theorems

The flux of a vector field $\vec{F}$ through an oriented surface S is $\int_S \vec{F} \cdot d\vec{S}$.

- If the surface S is the graph of a function $f(x, y)$, then

  $$\int_S \vec{F} \cdot d\vec{S} = \int_R \vec{F}\,(x, y, z(x, y)) \cdot \left[-\frac{\partial f}{\partial x}\vec{i} - \frac{\partial f}{\partial y}\vec{j} + \vec{k}\right]\, dx\, dy,$$

  where R is the domain of $f(x, y)$.

- If S is a parametric surface parametrized by $\vec{r} = \vec{r}(s,t)$ where s and t vary in a region R, then

$$\int_S \vec{F} \cdot d\vec{S} = \int_R \vec{F}\left(\vec{r}(s,t)\right) \cdot \left(\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}\right) \; ds \; dt,$$

  if we assume that $\dfrac{\partial \vec{r}}{\partial s} \times \dfrac{\partial \vec{r}}{\partial t}$ points in the direction of the normal $\vec{n}$ to the surface S everywhere.

Stokes's theorem links the circulation of a smooth vector field around a (piecewise) smooth closed curve C to the flux of curl$\vec{F}$ through any smooth surface S whose boundary is equal to C. It reads:

$$\int_C \vec{F} \cdot d\vec{r} = \int_S \text{curl}\vec{F} \cdot d\vec{A},$$

where the orientation of C is determined from the orientation of S (or vice-versa) by *the right hand-rule*. Note that Stokes's theorem is valid only if $\vec{F}$ **is defined everywhere** on S and C.

Green's theorem is a planar version of Stokes's theorem and reads

$$\int_C \vec{F} \cdot d\vec{r} = \int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \; dx \; dy,$$

where $\vec{F}(x,y) = F_1(x,y)\vec{i} + F_2(x,y)\vec{j}$ is a smooth vector field defined at every point of C as well as inside C, and C is a (simple) closed curve oriented such that its interior R is on the left as one moves along C.

5 The divergence theorem

The divergence of a vector field $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ with continuous partial derivatives is given by

$$\text{div}\vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Note that it is a **scalar** quantity. It has the following properties:

- $\mathrm{div}(\nabla f) = \nabla^2 f = \Delta f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$.
- $\mathrm{div}(f\nabla g) = f\nabla^2 g + \nabla f \cdot \nabla g$.
- $\mathrm{div}(f\vec{g}) = f\mathrm{div}(\vec{g}) + \nabla f \cdot \vec{g}$.
- The divergence of $\vec{F}$ at a point P is equal to the limit, when the surface S surrounding P is shrunk to zero, of the the flux of $\vec{F}$ through S (oriented outward) divided by the volume of the region W bounded by S:

$$\mathrm{div}\vec{F} = \lim_{W\to 0} \frac{\int_S \vec{F}\cdot d\vec{A}}{\text{volume of } W}.$$

- For any vector field $\vec{F}$ with continuous second partial derivatives,

$$\mathrm{div}\ \mathrm{curl}\vec{F} = \nabla\cdot(\nabla\times\vec{F}) = 0.$$

  Conversely, any smooth vector field $\vec{F}$ whose domain is closed and has no holes, and whose divergence is zero everywhere is a *curl field*, i.e. there exists a vector field $\vec{G}$ such that $\vec{F} = \mathrm{curl}\vec{G}$ (this is the *divergence test*).

The **divergence theorem** relates the flux of a smooth vector field through a (piecewise) smooth closed surface S oriented outward to the volume integral of its divergence over the region W bounded by S, and reads:

$$\int_S \vec{F}\cdot d\vec{A} = \int_W \mathrm{div}\vec{F}\ dV,$$

if $\vec{F}$ **is defined at every point** in W and on S.

Example: the continuity equation.

Consider a flow $\vec{v}(\vec{r})$ for a fluid of density $\rho(\vec{r},t)$. Call V a fixed region of the fluid domain. The mass of fluid in V is given by $m = \int_V \rho dV$, and its rate of change is

$$\frac{dm}{dt} = \int_V \frac{\partial\rho}{\partial t}\ dV,$$

if one assumes that ρ is smooth. On the other hand, this rate of change is given by the negative of the flux of matter $\rho\vec{v}$ through the boundary S of V, and by virtue of the divergence theorem,

$$\frac{dm}{dt} = -\int_S \rho\ \vec{v} \cdot d\vec{A} = -\int_V \mathrm{div}(\rho\ \vec{v})\ dV.$$

By comparison of the two expressions of dm/dt which are equal for every region V, we get the continuity equation, which reads

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho\ \vec{v}) = 0.$$