Math 583 A
Homework Assignment # 9

Due November 12, 1998

1 PDE's: Phase plane analysis

Consider the partial differential equation

$$u_t + u^2 u_x - u_{xxx} - u_{xx} = 0, \tag{1}$$

where u is real.

1. Make the traveling wave reduction $\xi = x - ct$ and find the ordinary differential equation satisfied by traveling wave solutions.

2. Integrate this equation once and use the boundary condition

   $$\lim_{\xi \to +\infty} u = \lim_{\xi \to +\infty} u_\xi = \lim_{\xi \to +\infty} u_{\xi\xi} = 0$$

   to find the constant of integration.

3. Analyze the two-dimensional dynamical system (for the variable ξ) found in 2., and draw the corresponding phase portraits (your answer should depend on c).

4. Identify special trajectories which would correspond to interesting traveling wave solutions of Eqn. (1). Sketch these solutions as functions of x, and indicate in which direction they travel.

2 PDE's: Linear stability analysis of simple solutions

Consider the complex Ginzburg-Landau equation

$$u_t = u + (1 + i\alpha)u_{xx} - (1 + i\beta)|u|^2 u, \tag{2}$$

where α and β are real, and u is complex.

1. Find the dispersion relation for the linearization of this equation about $u = 0$. Is the solution $u = 0$ linearly stable (assume all wavenumbers are permitted) ?

2. Find $R(k)$ and the nonlinear dispersion relation $\omega(k)$ for plane wave solutions of the form $u_k(x,t) = R(k)\exp[i(kx + \omega t)]$.

3. Linearize Eqn. (2) about the plane wave $u_k(x,t)$. Simplify your answer.

4. Set $k = 0$ and sketch the behavior of the eigenvalues of the linearized system as functions of the wavenumber q of the perturbation (again, assume all values of q are permitted).

   (a) Did you expect to find an eigenvalue equal to 0 at $q = 0$? Explain.

   (b) Find a condition on α and β for which u_0 is unstable, and draw the curve corresponding to this threshold condition in the parameter space. At a finite distance from this threshold, in the unstable region, a form of *weak turbulence* will develop, and the solution $u(x,t)$ of Eqn. (2) will display spatio-temporal disorder.

5. **Extra credit:** Find conditions on α and β for which u_k is linearly unstable. Show that if u_0 is linearly unstable, then all of the other plane waves u_k are also linearly unstable.