Math 583 A
Homework Assignment # 8

Due November 5, 1998

1 Contour integration

1. Prove that
$$\int_0^\infty \frac{\ln(1+x^2)}{1+x^2}\,dx = \pi \ln(2).$$
Hint: Consider the contour integral $\int_C \frac{\ln(i+z)}{1+z^2}\,dz$ on an appropriate contour C.

2. Evaluate $\int_0^\infty \frac{\ln^2(x)}{x^2+1}\,dx$.

3. Evaluate $\int_0^\infty \frac{\ln(x)}{x^2+1}\,dx$.

2 PDE's: Scalings, symmetries and phase plane analysis

1. Write the following partial differential equations in scaled form. Pay particular attention to the sign of each coefficient and take into account symmetries which would allow you to assume that the sign is fixed, without loss of generality.

   (a) The complex Ginzburg-Landay equation:
$$u_t = \mu u + (\alpha_r + i\alpha_i)u_{xx} - (\beta_r + i\beta_i)|u|^2 u,$$

where $\alpha_r > 0$ and $\beta_r > 0$, α_i and β_i are real, and u is complex.

(b) The nonlinear Schrödinger equation:

$$iu_t = Du_{xx} + \beta|u|^2u,$$

where D and β are real and u is complex.

(c) The Korteveg-de-Vries equation:

$$u_t = auu_x + bu_{xxx},$$

where a, b and u are real.

2. List all of the symmetries of the complex Ginzburg-Landau equation and of the nonlinear Schrödinger equation (defined above) written in scaled form.

3. For each of the partial differential equations below, draw the phase portrait associated with stationary solutions ($u_t = 0$).

   (a) $u_t = u_{xx} + u + u^3$
   (b) $u_t = u_{xx} + u - u^3$
   (c) $u_t = u_{xx} - u + u^3 - \gamma u^5$, $\gamma > 0$
   (d) $u_t = u_{xx} + u - u^3 + \gamma u^5$, $\gamma > 0$

4. For the linear diffusion equation

$$u_t = Du_{xx},$$

consider the solution $u = f(\lambda^\alpha x, \lambda^\beta t)$, where λ is the "stretching" parameter.

   (a) Find the α and β that leave the original equation invariant.
   (b) Find the "similarity" variable $z = z(x,t)$ that is invariant under the transformation $x \to \lambda^\alpha x$, $t \to \lambda^\beta t$.
   (c) Using this new variable, reduce the linear diffusion equation to an ordinary differential equation, i.e. set $u = f(z)$ where $z = z(x,t)$ found above.
   (d) Integrate this ordinary differential equation and discuss your result.