Math 583 A
Homework Assignment # 5

Due October 8, 1998

Analytic funtions - Part II

1. Let C be an open upper semicircle of radius R centered at the origin, and consider $\int_C f(z)\,dz$, where $f(z) = 1/(z^2+a^2)$, for real $a > 0$. Show that $|f(z)| \le 1/(R^2 - a^2)$, $R > a$, and
$$\left|\int_C f(z)\,dz\right| \le \frac{\pi R}{R^2 - a^2}, \qquad R > a.$$

2. We wish to evaluate the integral $I = \int_0^\infty e^{ix^2}\,dx$. Consider the contour C_R, which is the boundary of the circular sector in the upper half plane with vertices at 0, R and $Re^{i\pi/4}$.

   (a) Show that $I_R = \int_{C_R} e^{iz^2}\,dz = 0$, and that $\lim_{R\to\infty} I_1 = 0$, where $I_1 = \int_{C_R^1} e^{iz^2}\,dz$ and C_R^1 is the part of C_R between R and $Re^{i\pi/4}$. Hint: use $\sin(x) \ge 2x/\pi$ on $0 \le x \le \pi/2$.

   (b) By breaking I_R into three terms, deduce that
   $$\lim_{R\to\infty}\left(\int_0^R e^{ix^2}\,dx - e^{i\pi/4}\int_0^R e^{-r^2}\,dr\right) = 0.$$

   (c) Evaluate I, using the fact that $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

3. Evaluate the integrals $\int_C f(z)\,dz$, where C is the unit circle centered at the origin and $f(z)$ is given below:

   (a) $f(z) = \dfrac{\sin(z)}{z}$

(b) $f(z) = \dfrac{1}{(2z-1)^2}$

(c) $f(z) = \dfrac{e^z}{z}$

(d) $f(z) = e^{z^2}\left(\dfrac{1}{z^2} - \dfrac{1}{z^3}\right)$

4. Let $f(z)$ be analytic and nonzero in a region R. Show that $|f(z)|$ has a minimum value in R that occurs on the boundary.

5. Keener's book: # 6.2.15 page 278.

   Use the fact that a polynomial $P_n(z)$ has a factor $(z - z_k)$ if and only if $P_n(z_k) = 0$.