Math 583 A
Homework Assignment # 2

Due September 17, 1998

1 Adjoint of a matrix and the Fredholm alternative

1. If A is a linear mapping of a Euclidean space S into itself, show that $(A^*)^* = A$.

2. Consider a linear map A from $\mathbf{R}^n$ to $\mathbf{R}^m$. Based on the material discussed in class, show that the nullspace of A^T is the orthogonal complement of the column space of A in $\mathbf{R}^m$. For a map A of $\mathbf{R}^n$ into itself, show that the Fredholm alternative is a direct consequence of the above statement.

3. Consider the system of equations

$$\begin{cases} x_1 - x_2 = b_1 \\ x_2 - x_3 = b_2 \\ x_3 - x_1 = b_3 \end{cases}$$

   Since the left-hand-sides add to zero, this system is solvable if and only if $b_1 + b_2 + b_3 = 0$. How does this illustrate the Fredholm alternative ?

4. Keener's book: problem # 1.4.1, page 50.

2 Spectral theorem for self-adjoint mappings

1. Show that eigenvectors of a $n \times n$ complex matrix A corresponding to different eigenvalues are linearly independent.

2. Prove the spectral theorem for a self-adjoint mapping H using the following steps:

(a) Show that H has only real eigenvalues.

(b) Show that H has no generalized eigenvectors.

(c) Show that eigenvectors of H corresponding to different eigenvalues are orthogonal.

(d) Use the spectral theorem for square matrices to finish the proof.

3 The Minimax Principle

1. A self-adjoint linear mapping A of a linear vector space S into itself is called *positive* if $< x, Ax > > 0$ for all $x \neq 0$. If A and B are two self-adjoint mappings of S into itself, we write $A < B$ if $B - A$ is positive.

   Let A and B denote two self-adjoint mappings satisfying $A < B$. Denote the eigenvalues of A (resp. B) arranged in increasing order by $a_1 \leq a_2 \leq a_3 \ldots \leq a_n$ (resp. $b_1 \leq b_2 \leq b_3 \ldots \leq b_n$), where n is the dimension of S. Show that

$$a_i < b_i, \qquad i = 1, \ldots n.$$

2. Keener's book: problem # 1.3.3, page 50.

3. Illustrate the minimax principle on the following matrices

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$