Math 583 A
Homework Assignment # 10

Due December 3, 1998

1 Applications of Fourier Series

1. Show that $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.
2. Keener's book: # 2.2.12 page 97.

   Note: Use Fubini's theorem to show that $h(t)$ is in $\mathrm{L}^1([0, 2\pi])$ if f and g are in $\mathrm{L}^1([0, 2\pi])$.

2 Dirichlet Kernel and Gibbs's phenomenon

1. Show that $D_n(x) = \sum_{|k| \leq n} e^{2\pi i k x} = \frac{\sin[(2n+1)\pi x]}{\sin(\pi x)}$ for $x \neq 0$, and that $D_n(0) = 2n+1$.
2. Evaluate $\int_0^1 D_n(x)\, dx$ and show that $\int_0^1 \frac{\sin[(2n+1)\pi x]}{\sin(\pi x)}\, dx = 1$.
3. For a summable function f in $\mathrm{L}^1\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right)$, define $S_n(x) = \sum_{|k| \leq n} \hat{f}(k) e^{2\pi i k x}$, and show that $S_n(x) = \int_{-1/2}^{1/2} D_n(x-y)\, f(y)\, dy$.
4. Consider $f(x) = \left| \begin{array}{ll} -1 & -\frac{1}{2} \leq x < 0 \\ 1 & 0 \leq x < \frac{1}{2} \end{array} \right.$. This function is discontinuous at $x = 0$ and we want to compare it to its Fourier series near $x = 0$.

   Use a computer to plot $S_n(x)$ for $-\frac{1}{2} \leq x < \frac{1}{2}$. Note that $S_n(x)$ corresponds to the first n terms of the Fourier series of f. Does $S_n(x)$ give a good approximation of f ? Can you make the oscillations of S_n smaller by increasing n ?

This is called Gibbs's phenomenon. Below, we obtain an estimate of S_n and show that the oscillations cannot be made as small as one wants by increasing n. This phenomenon is a good illustration of the pointwise difference between a discontinuous function and its Fourier series.

5. Show that S_n for f defined in 4 above satisfies $S_n(x) = \int_{-x}^{x} D_n - \int_{-x+1/2}^{x+1/2} D_n$. (Use the fact that D_n is even).

6. Integrate $\int_{-x+1/2}^{x+1/2} D_n$ by parts and show that, for $|x| \leq \frac{1}{4}$,

$$\left| S_n - \int_{-x}^{x} D_n \right| = \left| \int_{-x+1/2}^{x+1/2} D_n \right| \leq \frac{C}{n},$$

where C is some constant.

7. Show that

$$\int_{-x}^{x} D_n(y)\, dy - \int_{-x}^{x} \frac{\sin[(2n+1)\pi y]}{\pi y}\, dy$$

is bounded by a constant multiple of $1/n$ for $|x| \leq \frac{1}{4}$.

Conclude that, for $|x| \leq \frac{1}{4}$, S_n is well approximated by

$$\int_{-x}^{x} \frac{\sin[(2n+1)\pi y]}{\pi y}\, dy = \frac{2}{\pi} \int_{0}^{\pi(2n+1)x} \frac{\sin(y)}{y}\, dy.$$

8. Plot $\sin(y)/y$ and conclude that $\frac{2}{\pi} \int_{0}^{\pi(2n+1)x} \frac{\sin(y)}{y}\, dy$ achieves its maximum at $x = 1/(2n+1)$. Use a calculator or a computer to approximate the peak value $\frac{2}{\pi} \int_{0}^{\pi} \frac{\sin(y)}{y}\, dy$, and explain how it substantiates Gibbs's phenomenon.

3 Distributions

1. Keener's book # 4.1.1.c page 177

2. Keener's book # 4.1.6 page 178

3. Keener's book # 4.1.8 page 178

4. Keener's book # 4.1.9 page 178