Math 583 A
Homework Assignment # 1

Due September 10, 1998

1 Vector Calculus

1. Compute the line integral of

$$\vec{F}(x,y) = (2xy - \pi y^2 \sin(\pi x) + 1)\vec{i} + (x^2 + 2y\cos(\pi x) + 1)\vec{j}$$

   over the curve $y = xexp(1 - x^2)$, $0 \leq x \leq 1$.

2. Find $\int_C \vec{F} \cdot d\vec{r}$, where C is the triangle with vertices $(0,0)$, $(0,1)$ and $(1,1)$, oriented counterclockwise, and $\vec{F} = xy\vec{i} + y^2\vec{j}$.

3. Find the flux integral of $\vec{F}(x,y,z) = x\vec{i} + y\vec{j} + z\vec{k}$ over the surface $z = xy + 1$ above the square $0 \leq x \leq 1$, $0 \leq y \leq 1$, oriented upward.

4. Consider the vector field

$$\vec{v} = K(x^2 + y^2 + z^2)^{-3/2}(x\vec{i} + y\vec{j} + z\vec{k}), \qquad K > 0.$$

   (a) Compute $\mathrm{div}(\vec{v})$.

   (b) Compute the flux of $\vec{v}$ through a closed surface that does not contain the origin.

   (c) Compute the flux of $\vec{v}$ through the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$.

5. Compute the flux of $\vec{F}(x,y,z) = x^2y^2z\vec{k}$ through the cone $z = \sqrt{x^2 + y^2}$ with $0 \leq z \leq R$, oriented downward.

6. Water in a bathtub has velocity vector field near the drain given by

$$\vec{F} = -\frac{y + xz}{(z^2+1)^2}\vec{i} - \frac{yz - x}{(z^2+1)^2}\vec{j} - \frac{1}{(z^2+1)^2}\vec{k}.$$

(a) The drain in the bathtub is a disk in the xy-plane with center at the origin and radius 1. Find the rate at which the water is leaving the bathtub.

(b) Find $\int_C \vec{G} \cdot d\vec{r}$, where C is the edge of the drain, oriented clockwise when viewed from above, and where

$$\vec{G} = \frac{1}{2}\left(\frac{y}{z^2+1}\vec{i} - \frac{x}{z^2+1}\vec{j} - \frac{x^2+y^2}{z^2+1}\vec{k}\right).$$

(c) Explain why your answers to parts (a) and (b) are equal.

2 Linear Algebra

1. For a linear vector space with a Euclidean structure, check that the induced norm is a norm.

2. Prove the Schwarz inequality.