Problems for Quiz 14

Math 322. Spring, 2007.

1. Consider the initial value problem (IVP) defined by the partial differential equation (PDE)

$$u_t = u_{xx} - 2u_x + u, \qquad 0 < x < 1, \;\; t > 0 \tag{1}$$

with boundary conditions

$$u(0,t) = 0, \qquad u(1,t) = 0, \tag{2}$$

and initial condition

$$u(x,0) = f(x). \tag{3}$$

You will use the method of separation of variables to find the solution to this problem.

   (a) Look for a solution of the PDE of the form $u(x,t) = F(x)G(t)$ and set up the corresponding eigenvalue problems (Hint: You should use the boundary conditions (2) to set up the eigenvalue problem for $F(x)$).

   (b) Consider the eigenvalue problem for $F(x)$ that you found in part a). Is it in Sturm-Liouville form? Can you transform it into Sturm-Liouville form? (Hint: Use problem 6 of problem set 5.7 in your text book).

   (c) Consider again the eigenvalue problem for $F(x)$. Find the eigenvalues λ_n and the corresponding eigenfunctions $F_n(x)$.

   (d) Find the functions $G_n(t)$ corresponding to the eigenvalues λ_n that you found in part c) and write down explicit expressions for the solutions $u_n(x,t) = F_n(x)G_n(t)$ of the PDE.

   (e) Verify that the functions $u_n(x,t)$ that you found in part d) are indeed solutions of the PDE (1).

   (f) The solution to the IVP is obtained by the principle of superposition: $u(x,t) = \sum_n A_n u_n(x,t)$ where the constant coefficients A_n are chosen to satisfy the initial condition (3). Using your answer to part b) and your knowlege on orthogonal expansions arising from Sturm-Liouville problems, write an explicit expression for the coefficients A_n (your formula should involve $f(x)$).

   (g) Write the solution to the problem if

$$f(x) = 2e^x \sin(3\pi x) - e^x \sin(7\pi x).$$

2. Consider the boundary value problem (BVP) defined by Laplace's equation

$$u_{xx} + u_{yy} = 0 \qquad \text{on the square} \quad 0 < x, y < 2$$

subject to the boundary conditions

$$u(0,y) = 0, \qquad u(x,2) = 0, \qquad u(2,y) = 0, \qquad u(x,0) = 100\sin(\pi x/2).$$

Solve the BVP using the method of separation of variables.

3. Consider the initial value problem (IVP) defined by partial differential equation (PDE)

$$u_t = u_{xx} \qquad 0 \le x \le 2, \quad t \ge 0$$

subject to the boundary conditions

$$u(0,t) = 0, \qquad u(2,t) = 0$$

and the initial condition

$$u(x,0) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ 2 - x & \text{if } 1 \le x \le 2 \end{cases}.$$

Solve the IVP using the method of separation of variables.