$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

- Take the Fourier transform of the equation.

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\partial u}{\partial t} e^{-ihx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} c^2 \frac{\partial^2 u}{\partial x^2} e^{-ihx} dx$$

$$\downarrow$$

$$\frac{1}{\sqrt{2\pi}} \frac{d}{dt} \int_{-\infty}^{+\infty} u(x,t)\, e^{-ihx} dx = (ih)^2 \frac{c^2}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u(x,t) e^{-ihx} dx$$

$$\frac{d}{dt} \hat{u}(h) = -h^2 c^2 \hat{u}(h)$$

- Solve this equation : $\hat{u}(h) = C e^{-h^2c^2t} = \hat{u}(h)\Big|_{t=0} e^{-h^2c^2t}$

If the initial condition is

$u(x,0) = f(x)$, then $\hat{u}(h)\Big|_{t=0} = \hat{f}(h)$.

So $\hat{u}(k) = \hat{f}(k)\, e^{-k^2 c^2 t}$

- Take the inverse Fourier transform to get

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(k)\, e^{-k^2 c^2 t}\, e^{ikx}\, dk$$

Recall that $\mathcal{F}(e^{-ax^2})(k) = \frac{1}{\sqrt{2a}} e^{-k^2/4a}$

Let g be such that

$$\hat{g}(k) = e^{-k^2 c^2 t}$$

$$\frac{1}{4a} = c^2 t$$

$$\Rightarrow a = \frac{1}{4c^2 t}$$

So $\hat{g}(k) = \frac{1}{\sqrt{2a}} \sqrt{2a}\, e^{-k^2/4a}$

$$\Rightarrow g(x) = \sqrt{2a}\, e^{-ax^2}$$

Then $u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(k)\, \hat{g}(k)\, e^{ikx}\, dk$

So using the convolution theorem, we have that

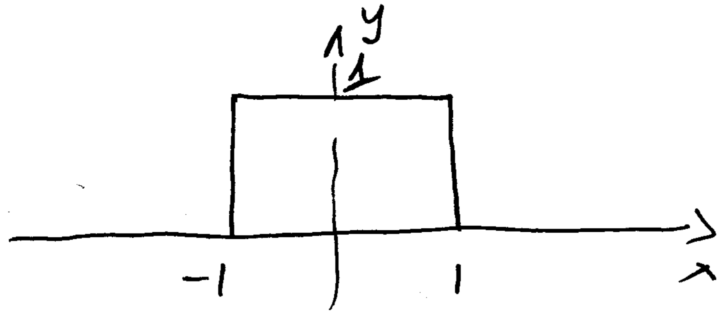
$$f * g = \sqrt{2\pi}\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(k)\, \hat{g}(k)\, e^{ikx}\, dx\right)$$

i.e. $$u(x,t) = \frac{1}{\sqrt{2\pi}} f * g$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x-t)\, g(t)\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y)\, g(x-y)\, dy$$

Example:



$$u(x,t) = \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{+\infty} f(y)\, e^{-\frac{(x-y)^2}{4c^2 t}}\, dy$$

$$= \frac{1}{2c\sqrt{\pi t}} \int_{-1}^{1} e^{-\frac{(x-y)^2}{4c^2 t}}\, dy.$$