The most general solution to

$$u_{xx} = c^2 \nabla^2 u$$

for a rectangular membrane and with Dirichlet boundary conditions is

$$u(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \, u_{mn}(x,y,t)$$

$$u_{mn}(x,y,t) = \left[\sin\left(n \frac{\pi x}{a}\right) \sin\left(m \frac{\pi y}{b}\right) \right]$$
$$\left[A_{mn} \cos\left(\sqrt{\left(\frac{m}{b}\right)^2 + \left(\frac{n}{a}\right)^2} \, \pi c t \right) + B_{mn} \sin\left(\sqrt{\left(\frac{m}{b}\right)^2 + \left(\frac{n}{a}\right)^2} \, \pi c t \right) \right]$$

To find the coefficient in front of each mode, we use the initial conditions.

$$u(x,y,0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \, u_{mn}(x,y,0)$$

$$u(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn} A_{mn} \sin\left(n\frac{\pi x}{a}\right)\sin\left(m\frac{\pi y}{b}\right)$$

$= f(x,y) \rightarrow$ We need to expand f as a double Fourier series.

$$\langle f,g\rangle = \int_0^a \int_0^b f(x,y)\,\overline{g(x,y)}\,dy\,dx$$

$$\langle f,f\rangle = \int_0^a \int_0^b \left[\sin\left(\frac{m\pi x}{a}\right)\right]^2 \left[\sin\left(\frac{n\pi y}{b}\right)\right]^2 dy\,dx$$

$$= \left[\int_0^a \left[\sin\left(\frac{m\pi x}{a}\right)\right]^2 dx\right]\left[\int_0^b \sin^2\left(\frac{n\pi y}{b}\right) dy\right]$$

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$
$$= 1 - \sin^2(\theta) - \sin^2(\theta) = 1 - 2\sin^2(\theta)$$

$$\Rightarrow \sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$$

$$\int_0^a \sin^2\left(\frac{m\pi x}{a}\right) dx = \int_0^a \frac{1-\cos\left(\frac{2m\pi x}{a}\right)}{2}\, dx$$

③

$$\left[\frac{x}{2} - \frac{1}{2}\,\frac{a}{2m\pi}\sin\left(\frac{2m\pi x}{a}\right)\right]_0^a$$

$$= \frac{a}{2} = \int_0^a \sin^2\left(\frac{m\pi x}{a}\right) dx$$

$$\langle f, f\rangle = \underbrace{\int_0^a \sin^2\left(\frac{m\pi x}{a}\right) dx}_{a/2}\ \underbrace{\int_0^b \sin^2\left(\frac{n\pi y}{b}\right) dy}_{b/2}$$

$$= \frac{ab}{4}$$

$$\frac{d^2F}{dx^2} = k\,F \qquad \text{boundary cond.}\quad F_x(0) = F_x(L) = 0$$

$$F(x) = A\,e^{\sqrt{k}\,x} + B\,e^{-\sqrt{k}\,x}$$

$$F'(x) = \sqrt{k}\,A\,e^{\sqrt{k}x} - B\sqrt{k}\,e^{-\sqrt{k}\,x}$$

$F'(0) = 0$ reads $\sqrt{k}A - B\sqrt{k} = 0 \Rightarrow A = B$

So $F(x) = A\left(e^{\sqrt{k}\,x} + e^{-\sqrt{k}\,x}\right)$

$$0 = F'(L) = \sqrt{k}\,A\,e^{\sqrt{k}L} - A\sqrt{k}\,e^{-\sqrt{k}L}$$

$$0 = e^{\sqrt{h'}L} - e^{-\sqrt{h'}L} \Rightarrow e^{2\sqrt{h'}L} = 1 = e^{2in\pi}$$

$$\Rightarrow 2\sqrt{h'}L = 2in\pi$$

$$\Rightarrow \sqrt{h} = \frac{in\pi}{L}$$

$$\Rightarrow h = -\left(\frac{n\pi}{L}\right)^2$$

Now, $F(x) = A\left(e^{\frac{in\pi}{L}x} + e^{-i\frac{n\pi x}{L}}\right)$

$$= 2A\cos\left(\frac{n\pi x}{L}\right)$$