Recall we found that

$$u_n(x,t) = a_n \cos\left(cn\frac{\pi t}{L}\right)\sin\left(n\frac{\pi x}{L}\right) + b_n \sin\left(cn\frac{\pi t}{L}\right)\sin\left(n\frac{\pi x}{L}\right)$$

for $n = 1, 2, \ldots$

solves the wave equation $u_{tt} = c^2 u_{xx}$.

Since the wave equation is linear, the most general solution that satisfies $u(0,t) = u(L,t) = 0$ is of the form

$$u(x,t) = \sum_{n=1}^{\infty} C_n u_n(x,t)$$

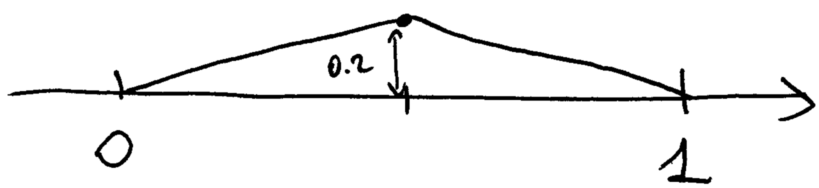
$$= \sum_{n=1}^{\infty} \left[A_n \cos\left(cn\frac{\pi t}{L}\right) + B_n \sin\left(cn\frac{\pi t}{L}\right)\right] \sin\left(n\frac{\pi x}{L}\right)$$

To find A_n & B_n, use the initial conditions.

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(n \frac{\pi x}{L}\right)$$

$$g(x) = \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} B_n \frac{c n \pi}{L} \sin\left(n \frac{\pi x}{L}\right)$$

Example:



① Since $\frac{\partial u}{\partial t}(x,0) = 0$, we have $g(x)=0$

So the B_n's are all 0.

② To find the A_n's :

Write that $A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$

We said $L = 1$

So $A_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx$

$$= 2\left[\int_0^{0.5} \frac{x}{5}\sin(n\pi x)\,dx + \int_{0.5}^{1} \frac{1-x}{5}\sin(n\pi x)\,d\right.$$

$$= \frac{2}{5}\left[\left[\frac{-\cos(n\pi x)}{n\pi}\,x\right]_0^{0.5} + \int_0^{0.5}\frac{\cos(n\pi x)}{n\pi}\,dx\right.$$

$$\left. + \left[(1-x)\,\frac{-\cos(n\pi x)}{n\pi}\right]_{0.5}^{1} - \int_{0.5}^{1}\frac{\cos(n\pi x)}{n\pi}\,dx\right]$$

$$= \frac{2}{5}\left[\left[\frac{\sin(n\pi x)}{(n\pi)^2}\right]_0^{0.5} - \left[\frac{\sin(n\pi x)}{(n\pi)^2}\right]_{0.5}^{1}\right]$$

$$= \frac{2}{5}\left[\frac{\sin(n\pi/2)}{(n\pi)^2} - \left[\frac{\sin(n\pi)}{(n\pi)^2} - \frac{\sin(n\pi/2)}{(n\pi)^2}\right]\right]$$

$$= \frac{4}{5}\,\frac{\sin(n\pi/2)}{n^2\pi^2}$$

2-d wave equation

$$u_{tt} = c^2 (u_{xx} + u_{yy})$$

$$u(x,y,t) = F(x,y)\, G(t)$$

Plug in:

$$\frac{d^2 G}{dt^2} F(x,y) = c^2 (F_{xx} G + F_{yy} G)$$

$$\underbrace{\frac{1}{c^2 G} \frac{d^2 G}{dt^2}}_{\text{function of } t} = \underbrace{\frac{1}{F} (F_{xx} + F_{yy})}_{\text{function of } x \,\&\, y}$$

$$= k$$

Find k by applying boundary conditions to F,

$$F_{xx} + F_{yy} = k F$$

Separate variables again:

$$F(x,y) = H_1(x)\, H_2(y)$$

Plug in:

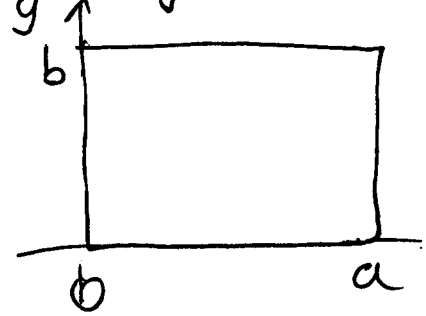
$$\frac{d^2H_1}{dx^2}\cdot H_2 + H_1\frac{d^2H_2}{dy^2} = k\,H_1H_2$$

$$\frac{1}{H_1}\frac{d^2H_1}{dx^2} + \frac{1}{H_2}\frac{d^2H_2}{dy^2} = k$$

i.e. $$\underbrace{\frac{1}{H_1}\frac{d^2H_1}{dx^2}}_{\text{function of } x} = \underbrace{k - \frac{1}{H_2}\frac{d^2H_2}{dy^2}}_{\text{function of } y}.$$

$$= Q$$

Rectangular membrane: $u(x,y)=0$ on the boundary of the rectangle $0 \leq x \leq a,\ 0 \leq y \leq b$



Solve $$\frac{d^2H_1}{dx^2} = QH_1$$
with boundary conditions $H_1(0)=H_1(a)=0$

Use results from last time

$$\frac{d^2F}{dx^2} = k F \quad \text{with } F(0) = F(L) = 0$$

gives $F(x) = a_n \sin\left(\frac{n\pi x}{L}\right)$ & $k = -\left(\frac{n\pi}{L}\right)^2$

Here, we get

$$H_1(x) = a_n \sin\left(n\frac{\pi x}{a}\right) \quad Q = -\left(\frac{n\pi}{a}\right)^2$$

Then go back to the equation for y:

$$\underbrace{k - Q}_{P} = \frac{1}{H_2}\frac{d^2H_2}{dy^2}$$

with boundary conditions: $H_2(0) = H_2(b) = 0$

Use (again) the results from last time

$$H_2(y) = b_m \sin\left(m\frac{\pi y}{b}\right) \quad P = -\left(\frac{m\pi}{b}\right)^2$$

$$P = k - Q \Rightarrow k = P + Q$$

$$= -\left(\frac{m\pi}{b}\right)^2 - \left(\frac{n\pi}{a}\right)^2$$

So $F(x,y) = C_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right)$

with $k = -\left[\left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{a}\right)^2\right]$

$m, n = 1, 2, 3, \ldots$

Go back to G:

$$\frac{1}{c^2 G} \frac{d^2 G}{dt^2} = k = -\left[\left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{a}\right)^2\right]$$

$$G(t) = A \cos\left(\sqrt{\left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{a}\right)^2}\, c\, t\right)$$

$$+ B \sin\left(\sqrt{\left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{a}\right)^2}\, c\, t\right)$$