4/12/07

(1)

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

u is a solution

$\Rightarrow$ αu is also a solution

α = constant

$$\frac{\partial^2 (\alpha u)}{\partial t^2} = \alpha \frac{\partial^2 u}{\partial t^2}$$

$$= \alpha c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$= c^2 \left(\frac{\partial^2 (\alpha u)}{\partial x^2} + \frac{\partial^2 (\alpha u)}{\partial y^2}\right)$$

If u_1 & u_2 are solutions, then $u_1 + u_2$ is also a solution.

$$u_{tt} = c^2 u_{xx}$$

$$u(x,t) = f(x+ct)$$

$$\frac{\partial u}{\partial t} = f'(x+ct) \cdot c$$

$$\frac{\partial^2 u}{\partial t^2} = \left[f''(x+ct) \cdot c\right] c$$

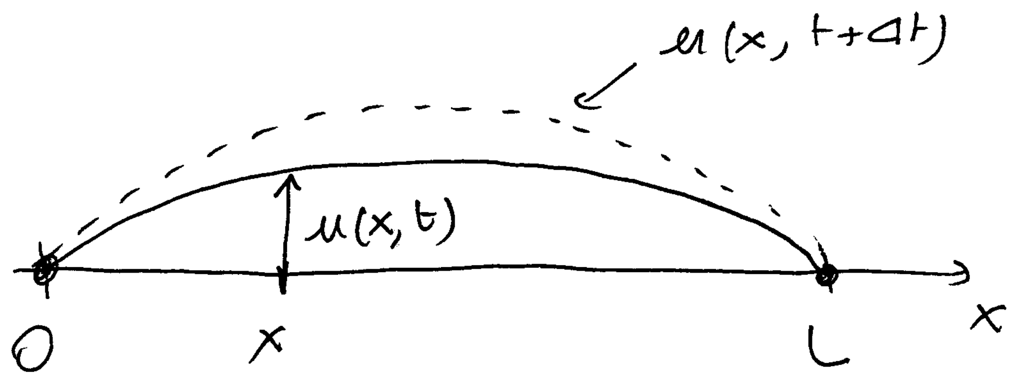
$$= c^2 f''(x+ct)$$

$$\frac{\partial u}{\partial x} = f'(x+ct) \cdot 1 = f'(x+ct)$$

$$\frac{\partial^2 u}{\partial x^2} = f''(x+ct)$$

$u_{tt} = c^2 u_{xx}$ reads

$$c^2 f''(x+ct) = c^2 f''(x+ct) \checkmark$$



$$u_{tt} = c^2 u_{xx}$$

1 $\quad u(x,t) = F(x)\,G(t)$

$$\frac{\partial u}{\partial t} = F(x)\frac{dG}{dt} \qquad \frac{\partial^2 u}{\partial t^2} = F(x)\frac{d^2G}{dt^2}$$

$$\frac{\partial u}{\partial x} = \frac{dF}{dx}\,G(t) \qquad \frac{\partial^2 u}{\partial x^2} = \frac{d^2F}{dx^2}\,G(t)$$

$u_{tt} = c^2 u_{xx}$ becomes

$$\underbrace{c^2\frac{d^2F}{dx^2}\,G(t)}_{c^2 u_{xx}} = \underbrace{F\frac{d^2G}{dt^2}}_{u_{tt}}$$

Divide by $c^2 G(t) F(x)$ to get

$$\frac{1}{F(x)}\frac{d^2F}{dx^2} = \frac{1}{c^2 G(t)}\frac{d^2G}{dt^2} = k$$

$A(x) = B(t)$ for all x, t

Choose x and get $A(x) = 3$

2. Solve the equation for x

$$\frac{1}{F}\frac{d^2F}{dx^2} = k$$

i.e. $$\frac{d^2F}{dx^2} = kF$$

Solution $F(x) = Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x}$

Boundary conditions:

$$u(x,t) = F(x)\,G(t)$$

$$u(0,t) = u(L,t) = 0$$

i.e. $F(0)\,G(t) = F(L)\,G(t) = 0$

Assume $u(x,t) \neq 0$ so that $F(0) = F(L) = 0$

Impose boundary conditions on F:

$$F(x) = A e^{\sqrt{k}x} + B e^{-\sqrt{k}x}$$

$$F(0) = A + B = 0 \Rightarrow B = -A$$

So $F(x) = A\left(e^{\sqrt{k}x} - e^{-\sqrt{k}x}\right)$

$$F(L) = A\left(e^{\sqrt{k}L} - e^{-\sqrt{k}L}\right) = 0$$

Since again we don't want $u(x,t) = 0$, we must choose k such that

$$e^{\sqrt{k}L} - e^{-\sqrt{k}L} = 0$$

i.e. $e^{\sqrt{k}L} = e^{-\sqrt{k}L} = \dfrac{1}{e^{\sqrt{k}L}}$

i.e. $e^{2\sqrt{k}L} = 1 = e^{i2\pi n}$

i.e. $e^{\sqrt{k}L} = e^{in\pi}$

i.e. $\sqrt{k} = i\dfrac{n\pi}{L} \Rightarrow k = -\left(\dfrac{n\pi}{L}\right)^2$

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$$e^{\sqrt{h}L} = -e^{in\pi} = \left(e^{i\pi}\right)e^{in\pi}$$
$$= e^{i(n+1)\pi}$$

With $F(x) = A\left(e^{\sqrt{h}x} - e^{-\sqrt{h}x}\right)$

and $\sqrt{h} = i\dfrac{n\pi}{L}$

we have

$$F(x) = A\left(e^{i\frac{n\pi x}{L}} - e^{-i\frac{n\pi x}{L}}\right)$$
$$= \underbrace{A\,2i}_{a_n}\sin\left(\frac{n\pi x}{L}\right)$$
$$= a_n \sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, \ldots$$

3 Go back to the equation for G

$$\frac{1}{c^2 G(t)}\frac{d^2G}{dt^2} = k = -\left(\frac{n\pi}{L}\right)^2 \qquad n=1,2,\ldots$$

i.e. $$\frac{d^2G}{dt^2} = -c^2\left(\frac{n\pi}{L}\right)^2 G$$

Characteristic polynomial:

$$\lambda^2 = -\left(c\frac{n\pi}{L}\right)^2$$

i.e. $$\lambda = \pm i\,\frac{cn\pi}{L}$$

$$G(t) = C_1 e^{i\frac{cn\pi}{L}t} + C_2 e^{-i\frac{cn\pi}{L}t}.$$

If we want G to be real, we must choose $C_2 = \overline{C_1}$.

Then we can re-write the solution as $D_1 \cos\left(\frac{cn\pi}{L}t\right) + D_2 \sin\left(\frac{cn\pi}{L}t\right)$ where $D_1, D_2 \in \mathbb{R}$

(4) Remember $u(x,t) = f(x)\,G(t)$

i.e. $$u(x,t) = a_n \sin\left(n\frac{\pi x}{L}\right)\left(D_1\cos\left(\frac{cn\pi}{L}t\right) + D_2\sin\left(\frac{cn\pi}{L}t\right)\right)$$

i.e. $u_n(x,t)$

$$= \alpha_n \sin\left(n \frac{\pi x}{L}\right) \cos\left(\frac{cn\pi}{L} t\right) + \beta_n \sin\left(n \frac{\pi x}{L}\right) \sin\left(\frac{cn\pi}{L} t\right)$$

for $n = 1, 2, \ldots$