Chapters 7-8: Linear Algebra

Sections 7.5, 7.8 & 8.1

1. Linear systems of equations

- A linear system of equations of the form

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1\\
&a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2\\
&\ldots\\
&a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m
\end{aligned}
$$

can be written in matrix form as $AX = B$, where

$$
A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
$$

Solution(s) of a linear system of equations

- Given a matrix A and a vector B, a solution of the system $AX = B$ is a vector X which satisfies the equation $AX = B$.
- If B is not in the column space of A, then the system $AX = B$ has no solution. One says that the system is not consistent. In the statements below, we assume that the system $AX = B$ is consistent.
- If the null space of A is non-trivial, then the system $AX = B$ has more than one solution.
- The system $AX = B$ has a unique solution provided $\dim(\mathcal{N}(A)) = 0$.
- Since, by the rank theorem, $\text{rank}(A) + \dim(\mathcal{N}(A)) = n$ (recall that n is the number of columns of A), the system $AX = B$ has a unique solution if and only if $\text{rank}(A) = n$.

Solution(s) of a linear system of equations (continued)

- A linear system of the form $AX = 0$ is said to be homogeneous.
- Solutions of $AX = 0$ are vectors in the null space of A.
- If we know one solution X_0 to $AX = B$, then all solutions to $AX = B$ are of the form

  $$X = X_0 + X_h$$

  where X_h is a solution to the associated homogeneous equation $AX = 0$.
- In other words, the general solution to the linear system $AX = B$, if it exists, can be written as the sum of a particular solution X_0 to this system, plus the general solution of the associated homogeneous system.

2. Inverse of a matrix

- If A is a square $n \times n$ matrix, its inverse, if it exists, is the matrix, denoted by A^{-1}, such that

$$A\,A^{-1} = A^{-1}\,A = I_n,$$

where I_n is the $n \times n$ identity matrix.

- A square matrix A is said to be singular if its inverse does not exist. Similarly, we say that A is non-singular or invertible if A has an inverse.

- The inverse of a square matrix $A = [a_{ij}]$ is given by

$$A^{-1} = \frac{1}{\det(A)}\,[C_{ij}]^T\,,$$

where $\det(A)$ is the determinant of A and C_{ij} is the matrix of cofactors of A.

Determinant of a matrix

- The determinant of a square $n \times n$ matrix $A = [a_{ij}]$ is the scalar

$$\det(A) = \sum_{i=1}^{n} a_{ij} C_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij}$$

where the cofactor C_{ij} is given by

$$C_{ij} = (-1)^{i+j} M_{ij},$$

and the minor M_{ij} is the determinant of the matrix obtained from A by "deleting" the i-th row and j-th column of A.

- **Example:** Calculate the determinant of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

Properties of determinants

- If a determinant has a row or a column entirely made of zeros, then the determinant is equal to zero.
- The value of a determinant does not change if one replaces one row (resp. column) by itself plus a linear combination of other rows (resp. columns).
- If one interchanges 2 columns in a determinant, then the value of the determinant is multiplied by -1.
- If one multiplies a row (or a column) by a constant C, then the determinant is multiplied by C.
- If A is a square matrix, then A and A^T have the same determinant.

Properties of the inverse

- Since the inverse of a square matrix A is given by

$$A^{-1} = \frac{1}{\det(A)} \left[C_{ij}\right]^T ,$$

we see that A is invertible if and only if $\det(A) \neq 0$.

- If A is an invertible 2×2 matrix, $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix},$$

and $\det(A) = a_{11}a_{22} - a_{21}a_{12}$.

- If A and B are invertible, then

$$(AB)^{-1} = B^{-1}A^{-1} \qquad \text{and} \qquad \left(A^{-1}\right)^{-1} = A.$$

Linear systems of n equations with n unknowns

- Consider the following linear system of n equations with n unknowns,

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\ldots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

- This system can be also be written in matrix form as $AX = B$, where A is a square matrix.

- If $\det(A) \neq 0$, then the above system has a unique solution X given by

$$X = A^{-1}B.$$

Linear systems of equations - summary

Consider the linear system $AX = B$ where A is an $m \times n$ matrix.

- The system may not be consistent, in which case it has no solution.
- To decide whether the system is consistent, check that B is in the column space of A.
- If the system is consistent, then
  - Either $\text{rank}(A) = n$ (which also means that $\dim(\mathcal{N}(A)) = 0$), and the system has a unique solution.
  - Or $\text{rank}(A) < n$ (which also means that $\mathcal{N}(A)$ is non-trivial), and the system has an infinite number of solutions.

Linear systems of equations - summary (continued)

Consider the linear system $AX = B$ where A is an $m \times n$ matrix.

- If $m = n$ and the system is consistent, then
  - Either $\det(A) \neq 0$, in which case $\text{rank}(A) = n$, $\dim(\mathcal{N}(A)) = 0$, and the system has a unique solution;
  - Or $\det(A) = 0$, in which case $\dim(\mathcal{N}(A)) > 0$, $\text{rank}(A) < n$, and the system has an infinite number of solutions.
- Note that when $m = n$, having $\det(A) = 0$ means that the columns of A are linearly dependent.
- It also means that $\mathcal{N}(A)$ is non-trivial and that $\text{rank}(A) < n$.

3. Eigenvalues and eigenvectors

- Let A be a square $n \times n$ matrix. We say that X is an eigenvector of A with eigenvalue λ if

$$X \neq 0 \qquad \text{and} \qquad AX = \lambda X.$$

- The above equation can be re-written as

$$(A - \lambda I_n)X = 0.$$

- Since $X \neq 0$, this implies that $A - \lambda I_n$ is not invertible, i.e. that $\det(A - \lambda I_n) = 0$.

- The eigenvalues of A are therefore found by solving the characteristic equation $\det(A - \lambda I_n) = 0$.

Eigenvalues

- The characteristic polynomial $\det(A - \lambda I_n)$ is a polynomial of degree n in λ. It has n complex roots, which are not necessarily distinct from one another.
- If λ is a root of order k of the characteristic polynomial $\det(A - \lambda I_n)$, we say that λ is an eigenvalue of A of algebraic multiplicity k.
- If A has real entries, then its characteristic polynomial has real coefficients. As a consequence, if λ is an eigenvalue of A, so is $\bar{\lambda}$.
- It A is a 2×2 matrix, then its characteristic polynomial is of the form $\lambda^2 - \lambda \operatorname{Tr}(A) + \det(A)$, where the trace of A, $\operatorname{Tr}(A)$, is the sum of the diagonal entries of A.

Eigenvalues (continued)

- **Examples:** Find the eigenvalues of the following matrices.
  - $A = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$.
  - $B = \begin{bmatrix} -1 & 9 \\ 0 & 5 \end{bmatrix}$.
  - $C = \begin{bmatrix} -13 & -36 \\ 6 & 17 \end{bmatrix}$.
  - $D = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4 & -1 \\ -1 & 1 & 2 \end{bmatrix}$.

Eigenvectors

- Once an eigenvalue λ of A has been found, one can find an associated eigenvector, by solving the linear system

$$(A - \lambda I_n)\, X = 0.$$

- Since $\mathcal{N}(A - \lambda I_n)$ is not trivial, there is an infinite number of solutions to the above equation. In particular, if X is an eigenvector of A with eigenvalue λ, so is αX, where $\alpha \in \mathbb{R}$ (or $\mathbb{C}$) and $\alpha \neq 0$.

- The set of eigenvectors of A with eigenvalue λ, together with the zero vector, form a subspace of $\mathbb{R}^n$ (or $\mathbb{C}^n$), E_λ, called the eigenspace of A corresponding to the eigenvalue λ.

- The dimension of E_λ is called the geometric multiplicity of λ.

Eigenvectors (continued)

- **Examples:** Find the eigenvectors of the following matrices. Each time, give the algebraic and geometric multiplicities of the corresponding eigenvalues.
  - $A = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$.
  - $C = \begin{bmatrix} -13 & -36 \\ 6 & 17 \end{bmatrix}$.
  - $D = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4 & -1 \\ -1 & 1 & 2 \end{bmatrix}$.

Properties of eigenvalues and eigenvectors

- The geometric multiplicity m_λ of an eigenvalue λ is less than or equal to its algebraic multiplicity M_λ.
- If $M_\lambda = 1$, then $m_\lambda = 1$.
- If m_λ is not equal to M_λ, then one can find $M_\lambda - m_\lambda$ linearly independent generalized eigenvectors of A, by solving a sequence of equations of the form

$$(A - \lambda I_n)\, U_{i+1} = U_i, \qquad i \in \{1, \cdots, M_\lambda - m_\lambda\}$$

where $U_1 = X_\lambda$ is a genuine eigenvector of A with eigenvalue λ.

Properties of eigenvalues and eigenvectors (continued)

- **Examples:** Find the genuine and generalized eigenvectors of the following matrices
  - $M = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$
  - $N = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$
- If A has k distinct eigenvalues and $\mathcal{B}_1, \cdots, \mathcal{B}_k$ are bases of the corresponding generalized eigenspaces, then $\{\mathcal{B}_1, \cdots, \mathcal{B}_k\}$ is a basis of $\mathbb{R}^n$ (or $\mathbb{C}^n$).