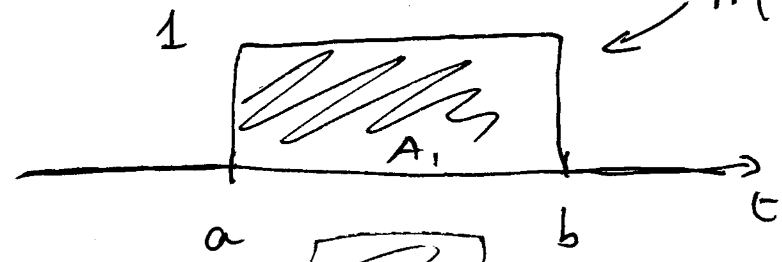
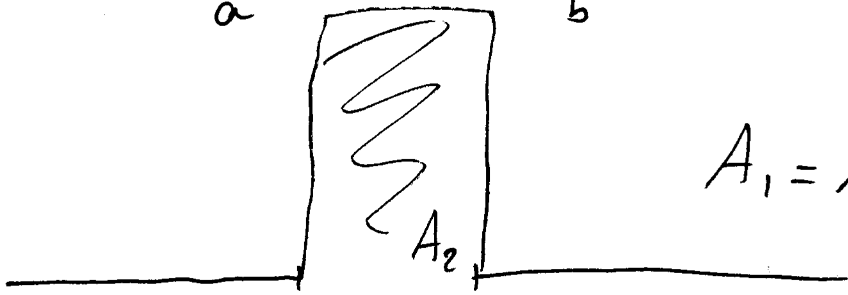
5/1/07

1

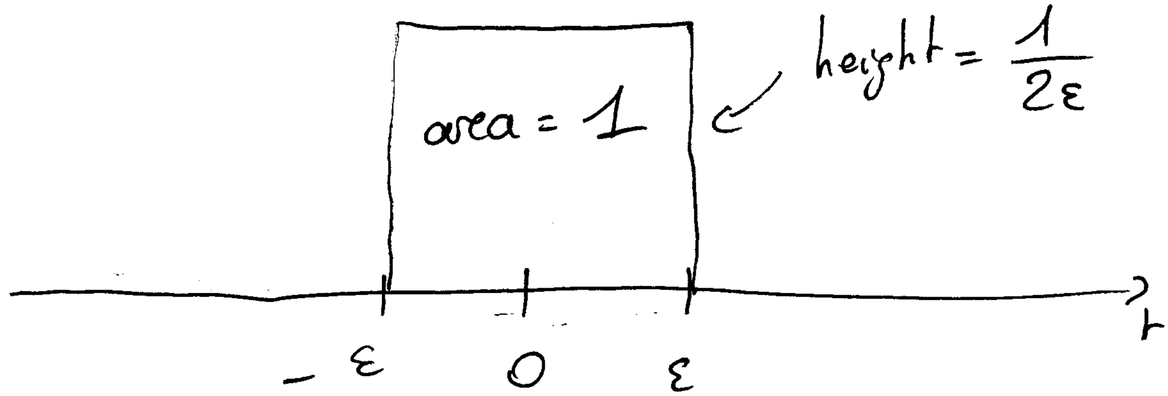
$H(t-a) - H(t-b)$



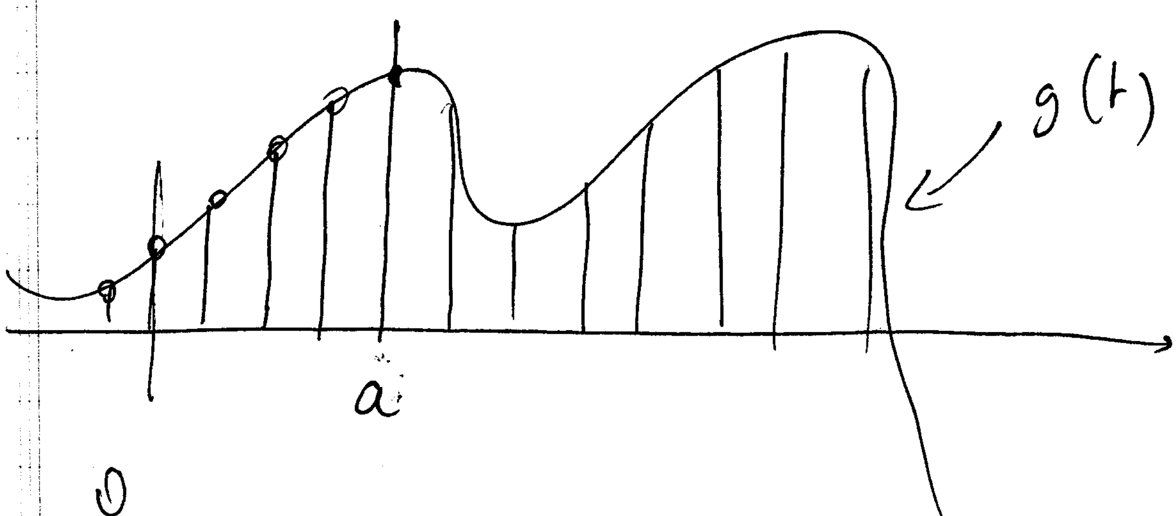
$A_1 = A_2$



height $= \frac{1}{2\varepsilon}$



$g(t)$



(2)

$$\mathcal{L}(\delta(t-a)) = \int_0^\infty e^{-st}\,\delta(t-a)\,dt = e^{-sa}$$

$$\mathcal{L}(t\,f(t))(s) = \int_0^\infty e^{-st}\,t\,f(t)\,dt$$

$$\mathcal{L}(f(t))(s) = \int_0^\infty e^{-st}\,f(t)\,dt$$

$$\frac{d}{ds}\left[\mathcal{L}(f(t))(s)\right] = \frac{d}{ds}\int_0^\infty e^{-st}\,f(t)\,dt$$

$$= \int_0^\infty \frac{d}{ds}\left(e^{-st} f(t)\right) dt$$

$$= \int_0^\infty -t\,e^{-st}\,f(t)\,dt$$

$$= -\int_0^\infty t\,e^{-st}\,f(t)\,dt$$

$$= -\mathcal{L}(t\,f(t))$$

$$\Rightarrow -\frac{d}{ds}\left[\mathcal{L}(f(t))(s)\right] = \mathcal{L}(t\,f(t))(s)$$

Let $G(s) = \int_s^\infty F(\nu)\, d\nu$ ← Laplace transform of f

$$= \int_s^\infty \int_0^\infty e^{-\nu t} f(t)\, dt\, d\nu$$

$$= \int_0^\infty f(t) \left(\int_s^\infty e^{-\nu t}\, d\nu \right) dt$$

$$= \int_0^\infty f(t) \left[\frac{e^{-\nu t}}{-t} \right]_s^\infty dt$$

$$= \int_0^\infty f(t) \frac{e^{-st}}{t}\, dt$$

$$= \mathcal{L}\left(\frac{f(t)}{t} \right)(s)$$

By inversion, $\frac{f(t)}{t} = \mathcal{L}^{-1}\left(\int_s^\infty F(\nu)\, d\nu \right)(t)$

<u>Example</u>: Find $u(s)$ such that

$$\frac{du}{ds} = \frac{s}{(s^2+1)^2}$$

$$\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{-2s}{(s^2+1)^2}$$

$$\frac{d}{ds}\left(\frac{-1/2}{s^2+1}\right) = \frac{s}{(s^2+1)^2}$$

But $\mathcal{L}(\sin(t)) = \dfrac{1}{1+s^2}$

So $$\frac{s}{(s^2+1)^2} = -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$

$$= -\frac{1}{2}\frac{d}{ds}\mathcal{L}\left(\sin(t)\right)(s)$$

So $$\mathcal{L}^{-1}\left(\frac{s}{(s^2+1)^2}\right) = \frac{1}{2}\mathcal{L}^{-1}\left(-\frac{d}{ds}\mathcal{L}(\sin(t))(s)\right)(t$$

$$= \frac{1}{2}t\sin(t)$$

<u>Example</u>: Solve $$\frac{d}{dt}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} -13 & -36 \\ 6 & 17 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{X}$$

$$X(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}.$$

"Old method":

1. Find the eigenvalues λ_1 & λ_2 of A

2. Find the eigenvectors U_1 & U_2

3. General solutions:

$$X = C_1 U_1 e^{\lambda_1 t} + C_2 U_2 e^{\lambda_2 t}$$

4. Find C_1 & C_2 by imposing the initial condition.

In terms of Laplace transforms:

$$\begin{cases} \dfrac{dx_1}{dt} = -13x_1 - 36x_2 \\ \dfrac{dx_2}{dt} = 6x_1 + 17x_2 \end{cases} \qquad X(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

Let $X_1(s) = \mathcal{L}(x_1(t))(s)$

$X_2(s) = \mathcal{L}(x_2(t))(s)$

Take the Laplace transform of each equation.

We get

$$\mathcal{L}\left(\frac{dx_1}{dt}\right) = \mathcal{L}\left(-13x_1 - 36x_2\right)$$

$$\Rightarrow s\mathcal{L}(x_1) - x_1(0) = -13\mathcal{L}(x_1) - 36\mathcal{L}(x_2)$$

$$\Rightarrow sX_1 - x_1(0) = -13X_1 - 36X_2$$

Similarly,

$$sX_2 - x_2(0) = 6X_1 + 17X_2$$

Therefore, we need to solve:

$$\begin{cases} (-13-s)X_1 - 36X_2 = -x_1(0) \\ 6X_1 + (17-s)X_2 = -x_2(0) \end{cases}$$

i.e. $$\underbrace{\begin{bmatrix} -13-s & -36 \\ 6 & 17-s \end{bmatrix}}_{B = A - sI} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} -x_1(0) \\ -x_2(0) \end{bmatrix}$$

So $$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = B^{-1} \begin{bmatrix} -x_1(0) \\ -x_2(0) \end{bmatrix}$$

So $B^{-1}\begin{bmatrix}-x_1(0)\\-x_2(0)\end{bmatrix} = \dfrac{1}{\det(B)}\begin{bmatrix}17-s & 36\\-6 & -13-s\end{bmatrix}\begin{bmatrix}-x_1(0)\\-x_2(0)\end{bmatrix}$

i.e.

$$\begin{pmatrix}X_1\\X_2\end{pmatrix} = \frac{1}{\det(B)}\begin{bmatrix}17-s & 36\\-6 & -13-s\end{bmatrix}\begin{bmatrix}-x_1(0)\\-x_2(0)\end{bmatrix}$$

$\det(B) = \det(A-sI)$ = characteristic polynomial of.

$$= \begin{vmatrix}-13-s & -36\\6 & 17-s\end{vmatrix}$$

$$= s^2 - Tr(A)s + \det(A)$$

$$= s^2 - 4s + (-13\cdot 17) + 6\cdot 36$$

$$= s^2 - 4s - 5 = (s-5)(s+1)$$

So $X_1 = \dfrac{-x_1(0)\,(17-s) - 36\,x_2(0)}{(s-5)\,(s+1)}$

$$X_2 = \frac{6\,x_1(0) + (13+s)\,x_2(0)}{(s-5)\,(s+1)}$$

Use partial fractions to re-write X_1 & X_2 in a form simple enough to take the inverse Laplace transform.

$$X_1 = \frac{\frac{-x_1(0)(17-5)-36x_2(0)}{6}}{s-5} + \frac{\frac{-x_1(0)(17+1)-36x_2(0)}{-6}}{s+1}$$

$$\left[\; X_1 = \frac{ax+b}{(x-5)(x+1)} = \frac{A}{x-5} + \frac{B}{x+1}\right.$$

$$\Rightarrow \frac{ax+b}{x+1} = A + B\,\frac{x-5}{x+1}$$

$$\left.\text{Set } x=5 \text{ to get } A = \frac{5a+b}{5+1}\;\right]$$

So $$X_1 = \frac{-2x_1(0)-6x_2(0)}{s-5} + \frac{3x_1(0)+6x_2(0)}{s+1}$$

$$X_2 = \frac{6x_1(0)+18x_2(0)}{(s-5)6} + \frac{6x_1(0)+12x_2(0)}{(s+1)(-6)}$$

$$= \frac{x_1(0)+3x_2(0)}{s-5} - \frac{x_1(0)+2x_2(0)}{s+1}$$

Take the Inverse Laplace transform to get:

$$x_1(t) = \left[-2x_1(0) - 6x_2(0)\right]e^{5t} + \left[3x_1(0) + 6x_2(0)\right]e^{-t}$$

$$x_2(t) = \left[x_1(0) + 3x_2(0)\right]e^{5t} - \left(x_1(0) + 2x_2(0)\right)e^{-t}$$

i.e.
$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \left(x_1(0)\begin{bmatrix} -2 \\ 1 \end{bmatrix} + 3\,x_2(0)\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)e^{5t} + \left(x_1(0)\begin{bmatrix} 3 \\ -1 \end{bmatrix} + 2\,x_2(0)\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right)e^{-t}$$

$$= \left[x_1(0) + 3x_2(0)\right]\begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{5t} + \left[x_1(0) + 2x_2(0)\right]\begin{bmatrix} 3 \\ -1 \end{bmatrix} e^{-t}$$

$$= C_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{5t} + C_2 \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^{-t}$$