4/26/07

1

Linear transformation

T is a linear transformation

if $T(\alpha f_1 + \beta f_2) = \alpha T(f_1) + \beta T(f_2)$.

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$\mathcal{L}(\alpha f_1 + \beta f_2)(s)$$

$$= \int_0^\infty e^{-st} \left[\alpha f_1(t) + \beta f_2(t)\right] dt$$

$$= \int_0^\infty e^{-st} \alpha f_1(t)\, dt + \int_0^\infty e^{-st} \beta f_2(t)\, dt$$

$$= \alpha \int_0^\infty e^{-st} f_1(t)\, dt + \beta \int_0^\infty e^{-st} f_2(t)\, dt$$

$$= \alpha \mathcal{L}(f_1)(s) + \beta \mathcal{L}(f_2)(s)$$

$$\mathcal{L}(e^{at}f(t))(s)$$

$$= \int_0^\infty e^{at} f(t)\, e^{-st}\, dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\, dt = \mathcal{L}(f)(s-a)$$

$$= F(s-a)$$

$$e^{at} f(t) = \mathcal{L}^{-1}(F(s-a))(t)$$

$$\mathcal{L}(f')(s) = \int_0^\infty f'(t)\, e^{-st}\, dt$$

$$= \left[f(t)\, e^{-st} \right]_0^\infty - \int_0^\infty f(t)\, (-s\, e^{-st})\, dt$$

$$= 0 - f(0) + s \int_0^\infty f(t) e^{-st}\, dt$$

$$= s\, \mathcal{L}(f)(s) - f(0)$$

$$\mathcal{L}(f'')(s) = \mathcal{L}((f')')(s)$$

$$= s\,\mathcal{L}(f')(s) - f'(0)$$

$$= s\left[s\,\mathcal{L}(f)(s) - f(0)\right] - f'(0)$$

$$= s^2\,\mathcal{L}(f)(s) - s f(0) - f'(0)$$

$$\mathcal{L}(f''')(s) = s\,\mathcal{L}(f'')(s) - f''(0)$$

$$= s^3\,\mathcal{L}(f)(s) - s^2 f(0) - s f'(0) - f''(0)$$

$$\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) = \int_0^\infty e^{-st} \int_0^t f(\tau)\,d\tau\,dt$$

Not obvious like this. We know that

$$\mathcal{L}(f')(s) = s\,\mathcal{L}(f(s)) - f(0)$$

Let us apply this formula to

$$g(t) = \int_0^t f(\tau)\,d\tau$$

$$g'(t) = f(t)$$

$$\mathcal{L}(g')(s) = s\mathcal{L}(g)(s) - g(0)$$

i.e.

$$\mathcal{L}(f(t))(s) = s\,\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) - \underbrace{g(0)}_{0}$$

$$= s\,\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right)$$

$$\Rightarrow \mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) = \frac{1}{s}\mathcal{L}(f(t))(s)$$

Apply $\mathcal{L}^{-1}$ to get

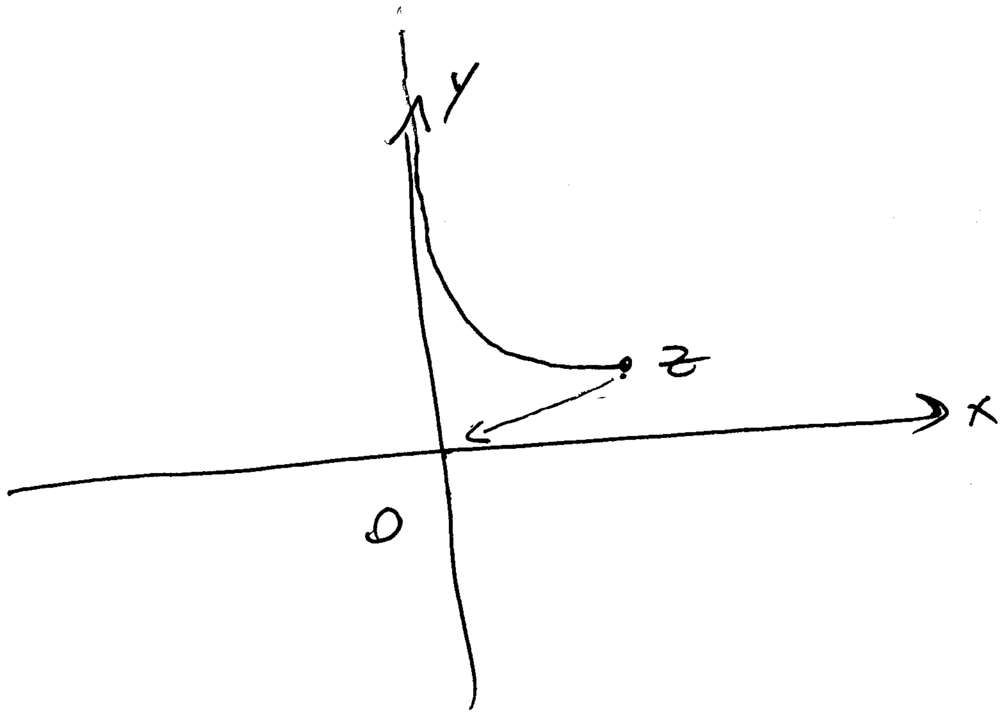
$$\int_0^t f(\tau)\,d\tau = \mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}(f)(s)\right)(t)$$

$$\mathcal{L}(\sin(\omega t))(s) = \int_0^\infty e^{-st}\sin(\omega t)\,dt$$

$$= \int_0^\infty e^{-st}\,\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\,dt$$

$$= \frac{1}{2i}\int_0^\infty \left[e^{(-s+i\omega)t} - e^{-(s+i\omega)t}\right]dt$$

(4')



$$|z| = \sqrt{x^2+y^2} = \text{distance from origin}$$

$$|e^{-(s+i\omega)t}| = |e^{-st}(\cos(\omega t) + i\sin(\omega t))|$$

$$= |e^{-st}|\, 1 = e^{-st} \xrightarrow[t\to\infty]{} 0$$

$$\mathcal{L}(\sin(\omega t))(s)$$
$$= \frac{1}{2i}\left(\left[\frac{e^{(-s+i\omega)t}}{-s+i\omega}\right]_0^\infty + \left[\frac{e^{-(s+i\omega)t}}{s+i\omega}\right]_0^\infty\right)$$
$$= \frac{1}{2i}\left[\frac{-1}{-s+i\omega} - \frac{1}{s+i\omega}\right]$$
$$= \frac{-1}{2i}\left[\frac{1}{s+i\omega} - \frac{1}{s-i\omega}\right]$$
$$= \frac{-1}{2i}\,\frac{s-i\omega-s-i\omega}{(s+i\omega)(s-i\omega)} = \frac{-1}{2i}\,\frac{-2i\omega}{s^2+\omega^2}$$
$$= \frac{\omega}{s^2+\omega^2}$$

$$\frac{d}{dt}\left[\sin(\omega t)\right] = \omega\cos(\omega t)$$

$$\mathcal{L}(\omega\cos(\omega t))(s) = s\,\mathcal{L}(\sin(\omega t)) - \sin(\omega t)\Big|_{t=0}$$
$$= s\,\frac{\omega}{\omega^2+s^2} - 0$$

i.e. $\omega\,\mathcal{L}(\cos(\omega t))(s) = \dfrac{s\omega}{\omega^2+s^2}$

i.e. $\mathcal{L}(\cos(\omega t))(s) = \dfrac{s}{\omega^2+s^2}$.

$$\mathcal{L}^{-1}\left(\frac{1}{s(s^2+1)}\right)$$

Recall

$$\mathcal{L}(\sin(\omega t)) = \frac{\omega}{\omega^2+s^2}.$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s}\,\mathcal{L}(\sin(t))\right)$$

$$= \int_0^t \sin(\tau)\,d\tau$$

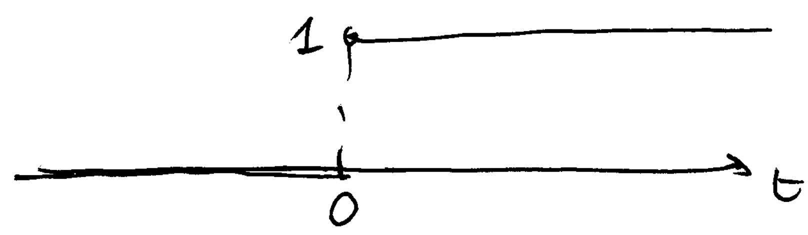
$$= \Big[-\cos(\tau)\Big]_0^t = -\cos(t) + 1$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+1)}\right) = \mathcal{L}^{-1}\left(\frac{1}{s}\,\mathcal{L}(1-\cos(t))\right)$$

$$= \int_0^t [1-\cos(\tau)]\,d\tau = \Big[\tau - \sin(\tau)\Big]_0^t$$

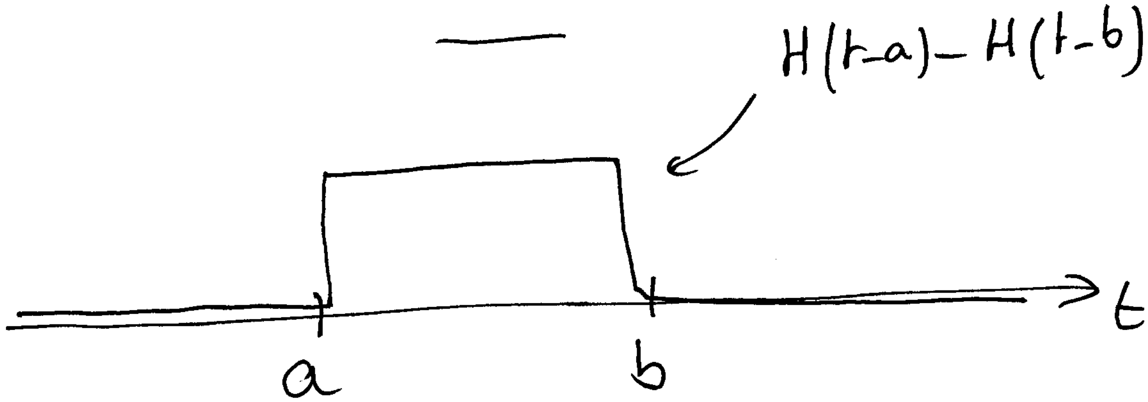
$$= t - \sin(t)$$

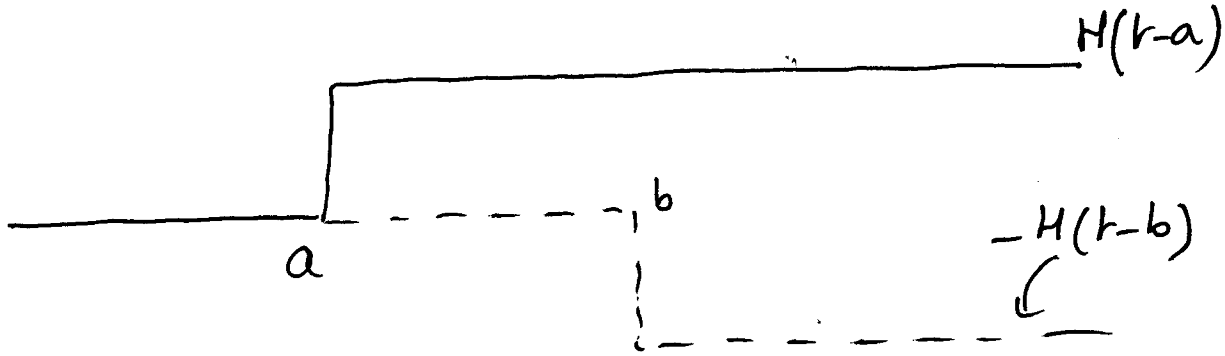
$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$



$$\mathcal{L}\big(H(t-a)\big) = \int_0^\infty H(t-a)\, e^{-st}\, dt$$

$$= \int_a^\infty e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_a^\infty = \frac{e^{-as}}{s}.$$





$$\mathcal{L}\big(f(t-a)H(t-a)\big) = \int_0^\infty f(t-a)H(t-a)\, e^{-st}\, dt$$

$$= \int_a^\infty f(t-a)\, e^{-st}\, dt \underset{y=t-a}{=} \int_0^\infty f(y)\, e^{-s(y+a)}\, dy$$

$$\mathcal{L}\big(f(t-a)H(t-a)\big)$$

$$= e^{-sa}\int_0^\infty f(y)e^{-sy}\,dy$$

$$= e^{-sa}\,\mathcal{L}(f)(s)$$

$$f(t-a)H(t-a) = \mathcal{L}^{-1}\left[e^{-as}\mathcal{L}(f(t))(s)\right](t)$$