Introduction to the mathematical modeling of multi-scale phenomena

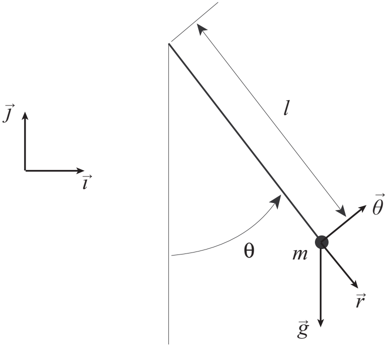
Scales & scalings

MCB/MATH 303

Scales & scalings

- The first lecture was concerned with multi-scale aspects of various physical or biological systems.
- We introduced the concept of scales and discussed examples of scale-free systems.
- In particular, we looked at examples of fractals and of self-similar systems found in nature.
- In this lecture, we consider how ideas of scales appear in models that involve differential equations.
- We start by introducing dimensional analysis. Then, we use this technique to make models dimensionless. Finally, we briefly discuss self-similar solutions to partial differential equations.

The nonlinear pendulum



Sketch of a point-mass pendulum

The equation of motion for the nonlinear pendulum is given by

$$m\, l\, \frac{d^2\theta}{dt^2} = -m\, g\, \sin(\theta) - c\, l\, \frac{d\theta}{dt},$$

where

- θ and t are variables.
- m, l, g and c are parameters.
- Most of these quantities are defined on the figure, except c, which measures friction.

Dimensions and dimensional analysis

Most quantities that appear in models have a dimension.

- T represents time. Quantities which have the dimension of a time may be expressed in different *units*, such as seconds, minutes, days, years, etc.
- M represents mass. The corresponding units may be grams, ounces, kilograms, etc.
- L represents length. Here again, such quantities my have different units, such as meters, feet, yards, kilometers, etc.
- To indicate that one considers the dimension of a quantity, one writes the symbol of that quantity between square brackets.

It is thus important to distinguish between dimension and units.

Dimensions and dimensional analysis (continued)

- For example, in the nonlinear pendulum equation

$$m\,l\,\frac{d^2\theta}{dt^2} = -m\,g\,\sin(\theta) - c\,l\,\frac{d\theta}{dt},$$

  - $[m] = M$, $[t] = T$, $[l] = L$.
  - $[g] = LT^{-2}$ and $[c] = MT^{-1}$.
  - θ is dimensionless, i.e. $[\theta] = 1$.

- Note that all of the terms in the equation have the same dimension.

- The number of significant parameters in a model can often be considerably reduced by introducing dimensionless variables.

Rescaling the equation for the nonlinear pendulum

- Define the following time scale $t_0 = \sqrt{l/g}$, and an associated dimensionless time variable τ, such that $\tau = \dfrac{t}{t_0}$.
- Then, make this change of variable in the nonlinear pendulum equation

$$m\, l\, \frac{d^2\theta}{dt^2} = -m\, g\, \sin(\theta) - c\, l\, \frac{d\theta}{dt},$$

- The resulting equation reads

$$\frac{d^2\theta}{d\tau^2} = -\sin(\theta) - \alpha \frac{d\theta}{d\tau}, \qquad \alpha = \frac{c}{m}\sqrt{\frac{l}{g}}.$$

- One can check that the parameter α is dimensionless.
- Note that this equation has only one parameter, α.

The classic SIR model

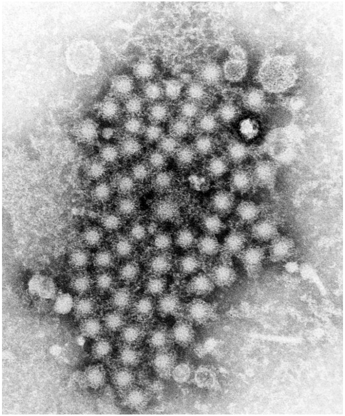
Penned goats in a village within a region investigated for a Rift Valley fever outbreak in Saudi Arabia. Picture # 8362, Public Health Image Library

The classic SIR model reads

$$\begin{aligned}\frac{dS}{dt} &= -\alpha\, S\, \frac{I}{N},\\ \frac{dI}{dt} &= \alpha\, S\, \frac{I}{N} - \beta\, I,\\ \frac{dR}{dt} &= \beta\, I,\end{aligned}$$

where S, I, and R represent the numbers of susceptible, infectious, and recovered (or removed) individuals, in a population of size N. The parameter α measures the average number of positive contacts per susceptible per unit of time, and β measures the rate at which individuals recover.

Viral infections



Transmission electron micrograph showing hepatitis virions of an unknown strain. Picture # 8153, Public Health Image Library

The the dynamics of a viral infection, such as hepatitis B or C, may be described by the following model (M.A. Nowak *et al.*, Proc. Natl. Acad. Sci. USA **93**, 4398-4402 (1996)).

$$\begin{aligned} \frac{dX}{dt} &= \lambda - \delta X - b\,V\,X \\ \frac{dY}{dt} &= b\,V\,X - a\,Y \\ \frac{dV}{dt} &= k\,Y - \kappa\,V \end{aligned}$$

The variable X represents the number of uninfected cells, Y is the number of infected cells, and V is the viral load (or number of free virions in the body).

Self-similar functions

- Assignment: do the homework problems on scalings and dimensional analysis. They are due in class on Thursday, September 11th.
- Recall that a self-similar object looks like itself over a range of scales.
- Self-similar structures found in nature are typically invariant over a finite number of magnifications.
- On the other hand, fractals are self-similar over an infinite number of magnifications.
- We now consider another example of self-similarity, this time in functions.

Self-similar functions (continued)

- Consider a function of two variables, $f(x,t)$, and assume that

  $$f(x,t) = \frac{1}{t^{\beta}}\, v\left(\frac{x}{t^{\alpha}}\right), \qquad \alpha,\ \beta \in \mathbb{R}$$

  where v is a given function of one variable.

- As t varies, one can think of $f(x,t)$ as describing the time-evolution of a function of x.

- The form of $f(x,t)$ defined above is such that if one multiplies x by t^{α} and f by t^{β}, then the resulting function remains unchanged as t varies.

- This is an example of self-similarity in a function, and is illustrated in the MATLAB GUI `Self_similar_function`.

Self-similar solution of the heat equation

- The heat equation is given by $\dfrac{\partial f}{\partial t} = D\,\dfrac{\partial^2 f}{\partial x^2}, \qquad D > 0.$
- We look for a solution in the form $f(x,t) = \dfrac{1}{t^\beta}\, v\left(\dfrac{x}{t^\alpha}\right)$, where $\alpha,\ \beta \in \mathbb{R}$.
- Note that since $z = \dfrac{x}{\sqrt{D\,t}}$ is dimensionless, it is natural to expect that v should be a function of z.
- Substituting the expression for $f(x,t)$ into the heat equation leads to the condition $2\alpha = 1$.
- Assuming $\alpha = \beta = 1/2$, the heat equation simplifies into $-v - z\dfrac{dv}{dz} = 2D\,\dfrac{d^2 v}{dz^2}$, where $z = \dfrac{x}{t^\alpha} = \dfrac{x}{\sqrt{t}}$.

Self-similar solution of the heat equation (continued)

- Solving this equation with the boundary conditions that v and its derivatives go to zero as $z \to \infty$ leads to

  $$v(z) = \kappa \exp\left(-\frac{z^2}{4D}\right),$$

  where κ is an arbitrary constant.

- In other words, we have found the following family of self-similar solutions to the heat equation,

  $$f(x,t) = \frac{\kappa}{\sqrt{t}} \exp\left(-\frac{x^2}{4D\,t}\right).$$

- This method clearly does not provide all solutions to a partial differential equation, but often gives useful insights into the behavior of the system modeled by the PDE in question.