

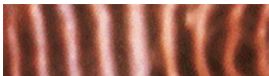
Introduction to the mathematical modeling of multi-scale phenomena

Overview

MCB/MATH 303

Pattern-forming systems

- Below are four **stripe patterns** found in nature. Taken out of context, and in the absence of color, they all look more or less the same. Can you recognize where they come from?



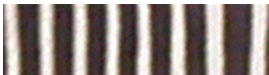
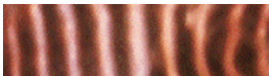
- Can you guess the **size** of each pattern?
- Can you discuss how many **scales** are visible in each of these patterns?

Pattern-forming systems (continued)

- Patterns are in fact **very common in nature and in the laboratory**.
- **Examples** include
 - Hexagonal and traveling wave patterns due to by **bioconvection**;
 - Spiral and target structures found in **slime moulds**, in **chemical reactions**, or in the **heart**;
 - Hexagonal geological structures such as **columnar joints**.
- Why do so many patterns **look alike**?
- The answer lies in a **universal theory of pattern formation**.

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Other examples of multi-scale systems

- **Hydrodynamic turbulence** observed in fluids is a typical example of a system that involves a variety of scales.
 - In **three dimensions**, energy cascades down to smaller scales until dissipation kicks in.
 - In **two dimensions**, such as in for instance **soap films**, an inverse cascade of energy is observed, leading to the formation of large-scale vortices.
- The following examples involve effects at the **micro- and nano-scales**.
 - The Lycurgus cup
 - The edelweiss
 - The gecko

This module will also discuss the following topics:

- **Self-similarity** and fractals.
- **Dimensional analysis** and scalings.
- Application to **self-similar solutions** of partial differential equations.
- **Diffusion** at the microscopic and macroscopic levels.
- Generalization of the above to **bacterial motion**.
- Time permitting, extension of these concepts to ideas of **anomalous diffusion** and **foraging behaviors**.

- Due **Thursday, August 28**: Quiz on plagiarism and self-plagiarism (D2L).
- Due **Tuesday, September 2**: Written essay (D2L).
- **Today in class**: identify one of the topics discussed above, to be presented in class on Tuesday, September 16.
- When you are done, start learning **how to use MATLAB**.

Two different types of “scale-free” systems

- **Pattern-forming systems**

- ① Pattern-forming systems display a **variety of periodic structure** (such as stripes, hexagons, squares, etc), which occur over a **wide range of scales**.
- ② The universal theory of pattern formation is somewhat **scale-free**, in the sense that it applies to phenomena that occur at different scales.
- ③ It is however built on the fact that **most patterns have at least two scales**, one which is **“fast”** (the period of the pattern), and one which is **“slow”** (the scale at which the pattern changes in space, or time).

- **Self-similar systems**

We now turn to another type of “scale free” systems, those which look identical at all scales. They are called **self-similar systems**.

Self-similarity and fractals

We will start our exploration of multi-scale systems with a discussion of self-similarity.

- **Self-similar** objects look identical at each scale (or magnification).
- **Fractals** are self-similar, as is illustrated in the *Julia Sets* MATLAB applet.
- We will spend some class time exploring and understanding **Julia sets**.

Diffusion-limited aggregation

- **Diffusion-limited aggregation** (DLA) is a model for the formation of aggregates proposed by Witten and Sander in 1981.
- In **two dimensions**, this process gives rise to branched clusters of fractal dimension $d = 1.71$.
- The process may be extended to **three dimensions**, where it gives rise to very complex clusters.

T.A. Witten & L.M. Sander, *Diffusion-limited aggregation, a kinetic critical phenomenon*, Phys. Rev. Lett. **47**, 1400-1403 (1981).

Fractals and artificial landscapes

- I am sure you all noticed that the three-dimensional DLA cluster looks like a tree.
- In fact, fractals are used to create **artificial landscapes** which look quite realistic.
- The picture on the right was created with the software **Teragen**.



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Fractals in nature and in the laboratory

- Objects found in nature, like the *Romanesco broccoli* pictured on the right, present self-similar structures.
- Self-similarity is also observed in laboratory experiments, such as the *tearing of a plastic sheet*.
- A simplified version of this experiment can be done in class.
- A direct *analogy* can be made between *torn plastic sheets* and some types of *leaves or petals*.



Image courtesy of A. Calini

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Image courtesy of A. Calini

The Lycurgus cup



These photographs, from [The British Museum's](#) web site, show the Lycurgus cup, made of dichroic glass. The cup changes color (from green to red) when it is held up to the light. This is due to colloidal particles of silver and gold imbedded into the glass.

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The edelweiss



Image by Daniel Schwen released under a Creative Commons Attribution ShareAlike 2.5 license

The velvety aspect of the edelweiss is due to the presence of filaments (of about $10\ \mu\text{m}$ in diameter) that protect the flower from damage due to UV light. These filaments are themselves covered with nano-scale fibers (of average diameter 180 nm).

K. Kertész, Zs. Bálint, Z. Vértesy, G.I. Márk, V. Lousse, J.-P. Vigneron and L.P. Biró, *Photonic crystal type structures of biological origin: Structural and spectral characterization*, *Current Appl. Phys.* **6**, 252-258 (2006).

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The gecko



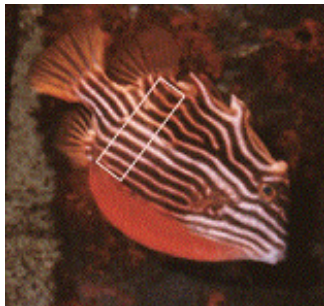
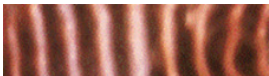
Image by Montrealais released under a [Creative Commons Attribution 3.0 Unported license](#)

The toes of a gecko are covered with micro-hairs which are themselves covered with possibly hundreds of nano-hairs. The structure of these hairs allows the gecko to climb on walls by means of dry adhesion.

The fabrication of synthetic gecko hair may lead to the commercialization of new types of residue-less adhesives. It is also used in the development of wall climbing robots (see for instance the Carnegie Mellon [NanoRobotics Lab](#) web site).

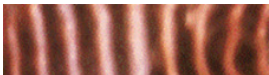
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Patterns in nature



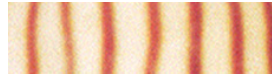
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Patterns in nature



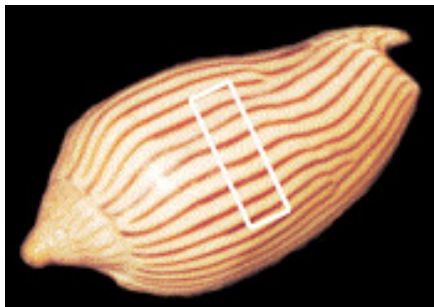
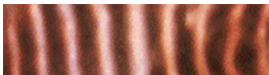
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Patterns in nature



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Patterns in nature



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Bioconvection

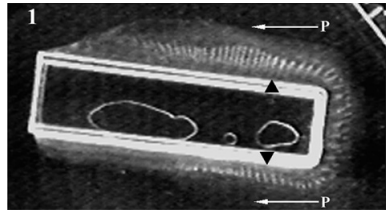
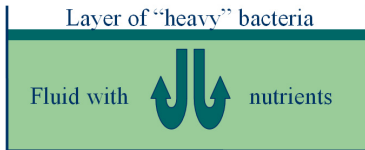


Image from N. Mendelson & J. Lega, *J. Bacteriology* **180**, 3285-3294 (1998).

- H. Wager, *On the effect of gravity upon the movements and aggregation of Euglena viridis, Ehrb., and other micro-organisms*, *Philos. Trans. R. Soc. London* **B 201**, 333-390 (1911).
- J.R. Platt, *Bioconvection patterns in cultures of free-swimming organisms*, *Science* **133**, 1766-1767 (1961).
- T. Pedley & J. Kessler, *Hydrodynamic phenomena in suspensions of swimming micro-organisms*, *Ann. Rev. Fluid Mech.* **24**, 313-358 (1992).
- N.A. Hill & T.J. Pedley, *Bioconvection*, *Fluid Dynamics Research* **37**, 1-20 (2005).