Review of Complex Numbers

Definitions, Algebra of complex numbers, Polar coordinates

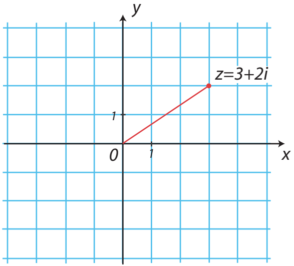
1. Complex numbers

- Complex numbers are of the form

$$z = x + iy, \qquad x, y \in \mathbb{R}, \qquad i^2 = -1.$$

- In the above definition, x is the real part of z and y is the imaginary part of z.

- The complex number $z = x + iy$ may be represented in the complex plane as the point with cartesian coordinates (x, y).



Complex conjugate

- The complex conjugate of $z = x + iy$ is defined as

$$\bar{z} = x - iy.$$

- As a consequence of the above definition, we have

$$\Re e(z) = \frac{z + \bar{z}}{2}, \qquad \Im m(z) = \frac{z - \bar{z}}{2i}, \qquad z\bar{z} = x^2 + y^2. \quad (1)$$

- If z_1 and z_2 are two complex numbers, then

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}, \qquad \overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}. \quad (2)$$

Modulus of a complex number

- The absolute value or modulus of $z = x + iy$ is

$$|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}.$$

  It is a positive number.

- **Examples:** Evaluate the following
  - $|i|$
  - $|2 - 3i|$

2. Algebra of complex numbers

- You should use the same rules of algebra as for real numbers, but remember that $i^2 = -1$.

  Examples:
  - Find powers of i and $1/i$.
  - Assume $z_1 = 2 + 3i$ and $z_2 = -1 - 7i$. Calculate $z_1 z_2$ and $(z_1 + z_2)^2$.

- Get used to writing a complex number in the form

$$z = (\text{real part}) + i\ (\text{imaginary part}),$$

  no matter how complicated this expression might be.

Algebra of complex numbers (continued)

- Remember that multiplying a complex number by its complex conjugate gives a real number.
- **Examples:** Assume $z_1 = 2 + 3i$ and $z_2 = -1 - 7i$.
  - Find z_1/z_2.
  - Find $\overline{z}_1/\overline{z}_2$.
  - Find $\Im m \left(1/\,\overline{z_1}^3\right)$.

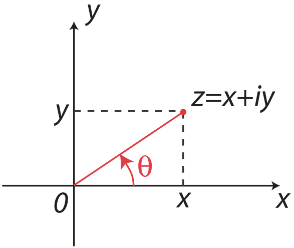
3. Polar coordinates form of complex numbers

- In polar coordinates,

$$x = r\cos(\theta), \qquad y = r\sin(\theta),$$

where

$$r = \sqrt{x^2 + y^2} = |z|.$$



- The angle θ is called the argument of z. It is defined for all $z \neq 0$, and is given by

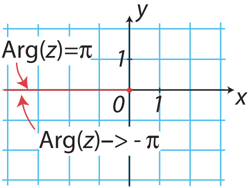
$$\arg(z) = \theta = \begin{cases} \arctan(y/x) & \text{if } x \geq 0 \\ \arctan(y/x) + \pi & \text{if } x < 0 \text{ and } y \geq 0 \\ \arctan(y/x) - \pi & \text{if } x < 0 \text{ and } y < 0 \end{cases} \pm 2n\pi.$$

Principal value $\mathrm{Arg}(z)$

- Because arg(z) is multi-valued, it is convenient to agree on a particular choice of arg(z), in order to have a single-valued function.
- The principal value of arg(z), $\mathrm{Arg}(z)$, is such that

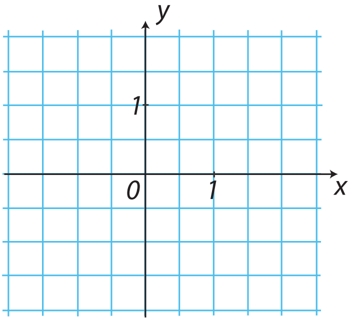
$$\tan\left(\mathrm{Arg}(z)\right) = y/x, \qquad \text{with } -\pi < \mathrm{Arg}(z) \leq \pi.$$

- Note that $\mathrm{Arg}(z)$ jumps by -2π when one crosses the negative real axis from above.



Principal value $\mathrm{Arg}(z)$ (continued)

- **Examples:**
  - Find the principal value of the argument of $z = 1 - i$.
  - Find the principal value of the argument of $z = -10$.



Polar and cartesian forms of a complex number

- You need to be able to go back and forth between the polar and cartesian representations of a complex number.

$$z = x + iy = |z|\cos(\theta) + i|z|\sin(\theta).$$

- In particular, you need to know the values of the sine and cosine of multiples of $\pi/6$ and $\pi/4$.
  - Convert $\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)$ to cartesian coordinates.
  - What is the cartesian form of the complex number such that $|z| = 3$ and $\mathrm{Arg}(z) = \pi/4$?

Euler's formula

- Euler's formula reads

$$\exp(i\theta) = \cos(\theta) + i\sin(\theta), \qquad \theta \in \mathbb{R}.$$

- As a consequence, every complex number $z \neq 0$ can be written as

$$z = |z|\left(\cos(\theta) + i\sin(\theta)\right) = |z|\exp(i\theta).$$

- This formula is extremely useful for calculating powers and roots of complex numbers, or for multiplying and dividing complex numbers in polar form.

Integer powers of a complex number

To find the n-th power of a complex number $z \neq 0$, proceed as follows

1. Write z in exponential form, $z = |z| \exp(i\theta)$.
2. Then take the n-th power of each side of the above equation

$$z^n = |z|^n \exp(i\, n\, \theta) = |z|^n \left(\cos(n\,\theta) + i \sin(n\,\theta)\right).$$

3. In particular, if z is on the unit circle ($|z| = 1$), we have

$$(\cos(\theta) + i \sin(\theta))^n = \cos(n\,\theta) + i \sin(n\,\theta).$$

This is De Moivre's formula.

Integer powers of a complex number (continued)

- **Examples of application:**
  - Trigonometric formulas

$$\begin{cases} \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta), \\ \sin(2\theta) = 2\sin(\theta)\cos(\theta). \end{cases} \tag{3}$$

  - Find $\cos(3\theta)$ and $\sin(3\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$.

Product of two complex numbers

- The product of $z_1 = r_1 \exp(i\theta_1)$ and $z_2 = r_2 \exp(i\theta_2)$ is

$$\begin{aligned} z_1 z_2 &= (r_1 \exp(i\theta_1))\ (r_2 \exp(i\theta_2)) \\ &= r_1 r_2 \exp(i(\theta_1 + \theta_2)) . \end{aligned} \tag{4}$$

- As a consequence,

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2), \qquad |z_1 z_2| = |z_1|\,|z_2|.$$

- We can use Equation (4) to show that

$$\begin{aligned} \cos(\theta_1 + \theta_2) &= \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) , \\ \sin(\theta_1 + \theta_2) &= \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) . \end{aligned} \tag{5}$$

Ratio of two complex numbers

- Similarly, the ratio z_1/z_2 of 2 complex numbers is given by

$$\frac{z_1}{z_2} = \frac{r_1 \exp\left(i\theta_1\right)}{r_2 \exp\left(i\theta_2\right)} = \frac{r_1}{r_2} \exp\left(i\left(\theta_1 - \theta_2\right)\right).$$

- As a consequence,

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2), \qquad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$$

- **Example:** Assume $z_1 = 2 + 3i$ and $z_2 = -1 - 7i$. Find $\left|\frac{z_1}{z_2}\right|$.

Roots of a complex number

To find the n-th roots of a complex number $z \neq 0$, proceed as follows (see the MIT applet for a graphic representation)

1. Write z in exponential form, $z = r\exp\left(i(\theta + 2p\pi)\right)$, with $r = |z|$ and $p \in \mathbb{Z}$.
2. Then take the n-th root (or the $1/n$-th power)

$$\sqrt[n]{z} = z^{1/n} = r^{1/n}\exp\left(i\frac{\theta + 2p\pi}{n}\right) = \sqrt[n]{r}\exp\left(i\frac{\theta + 2p\pi}{n}\right).$$

3. There are thus n roots of z, given by

$$z_k = \sqrt[n]{r}\left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right), \quad k = 0, \cdots, n-1.$$

Roots of a complex number (continued)

- The principal value of $\sqrt[n]{z}$ is the n-th root of z obtained by taking $\theta = \mathrm{Arg}(z)$ and $k = 0$.
- The n-th roots of unity are given by

$$\sqrt[n]{1} = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right) = \omega^k, \qquad k = 0, \cdots, n-1$$

  where $\omega = \cos(2\pi/n) + i\sin(2\pi/n)$.
- In particular, if w_1 is any n-th root of $z \neq 0$, then the n-th roots of z are given by

$$w_1,\ w_1\omega,\ w_1\omega^2, \cdots,\ w_1\omega^{n-1}.$$

Roots of a complex number (continued)

- **Examples:**
  - Find the three cubic roots of 1.
  - Find the four values of $\sqrt[4]{i}$.
  - Give a representation in the complex plane of the principal value of the eighth root of $z = -3 + 4i$.

Triangle inequality

- If z_1 and z_2 are two complex numbers, then

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

This is called the triangle inequality. Geometrically, it says that the length of any side of a triangle cannot be larger than the sum of the lengths of the other two sides.

- More generally, if z_1, z_2, ..., z_n are n complex numbers, then

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|.$$

Check your knowledge

Are you comfortable with the following algebraic manipulations?

- Addition, subtraction, multiplication, and division of complex numbers;
- Finding the real and imaginary parts and the modulus of combinations of complex numbers;
- Simplifying expressions that involve z and its complex conjugate.

Check your knowledge (continued)

Are you comfortable with the following algebraic manipulations?

- Going back and forth between polar and Cartesian representations of a complex number;
- Finding the argument, roots, logarithm, exponential, powers of a complex number.
- Given a complex number z, can you evaluate the principal value at z of each of the multi-valued functions we discussed in class?