Math 250A (Fall 2008) - Lab II (SOLUTIONS)

Part I

Consider the differential equation

$$\frac{dy}{dx} = y^2 + 1$$

(1) Solve this equation analytically to obtain an expression for $y(x)$. Your answer should contain an arbitrary constant.

Answer: A general solution can be found by first separating variables and then integrating both sides. One can then make a trigonometric substitution $y = \tan\theta$ and upon solving for y, the solution is

$$y(x) = \tan(x + C)$$

where C is a constant dependent upon the initial condition.

(2) Solve the differential equation numerically using Euler's method on the interval $x \in [0, 1]$ for the initial condition $y(0) = 0$. On a single figure, plot your estimated solution curve using the following step sizes for Δx: 0.5, 0.2, 0.1, 0.05, and 0.01. Make clear which curve corresponds to each step-size (Hint: use different line styles/colors).

Answer: An important component of the assignment was in this specific question. The main challenge stemmed from learning how to write a Matlab script that would do the numeric integration when executed. An example code is given below as well as the figure (Fig. 1) that was asked for. Make sure you understand how each line of the code works and how they ultimately all tie together to form the *program*.

```
% ### odeSOLVEptI.m ### 11.10.08

% Matlab code to use Euler's method to solve the differential equation
% y' = y^2 + 1

% NOTE: there are a lot of different ways (e.g., structure, syntax, naming
% conventions) one could use for this code that would still ultimately
% produce the same result

clear
clf

% ———————————————-
% User Input Parameters

% nsteps= 1000;       % # of steps
xMIN= 0;        % starting x-value
xMAX= 1;         % ending x-value
```
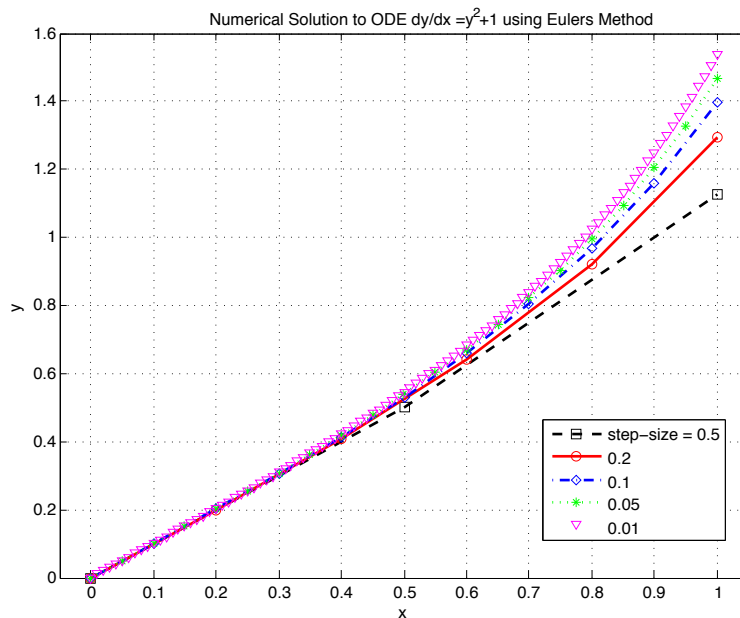
FIGURE 1. Comparison of numerical solutions for the differential equation $y' = y^2 + 1$ using Euler's method for different step-sizes.

```
deltaX= 0.01;       % step-size

% initial conditions (i.e. what is y(xMIN)?)
y0= 0;


% ———————————————-
% [user shouldn't have to change 'parameters' below here]

x0= xMIN;       % rename initial x-value

% determine # of steps needed based upon interval bounds and specified step-size
nsteps= (xMAX-xMIN)/deltaX;

% create the 'arrays' xS and yS that will contain all the x and y-values
% by setting first value equal to initial condition
yS(1)= y0; xS(1)= x0;

% use a for loop to go through each step
for nn=2:nsteps+1

    % use Euler's method to solve for next y-value
    % (this is where the form of the differential equation is input)
    yS(nn) = yS(nn-1) + deltaX*(yS(nn-1)^2 + 1);

    % update x-array
    xS(nn) = xS(nn-1)+ deltaX;

end

% plot the numerical solution
```
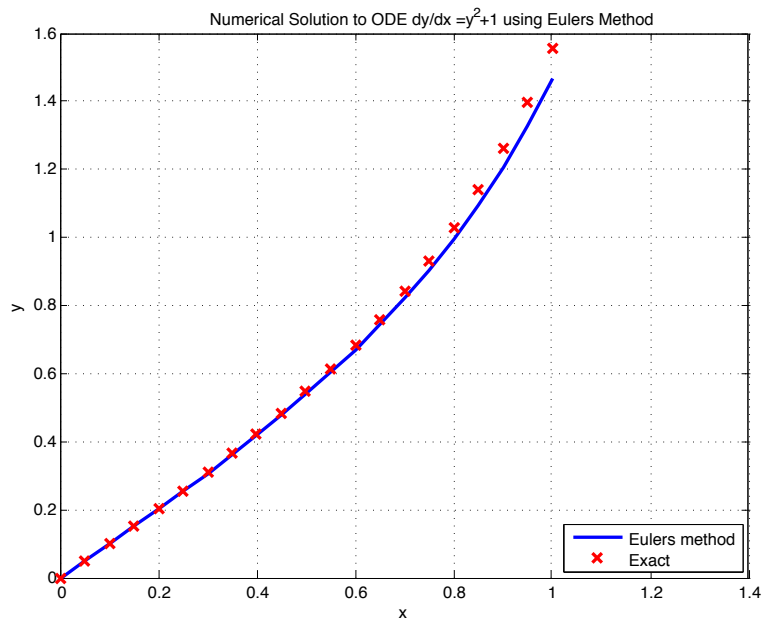
FIGURE 2. Comparison of an exact solution for the differential equation $y' = y^2 + 1$ with a solution with the same initial condition estimated numerically using Euler's method.

```
plot(xS,yS)
hold on;
xlabel('x')
ylabel('y')
title('Numerical Solution to ODE dy/dx =y^2+1 using Eulers Method')
% plot exact (i.e., analytically-derived) solution (only true if y(0)=0)
yA= tan(xS);
plot(xS,yA,'rx')
grid on
legend('Eulers method','Exact','Location','SouthEast')
```

(3) On a different figure, plot your numerical solution above for $\Delta x = 0.05$ together with the exact solution you solved for analytically (given the specified initial condition). Make sure that it is clear which of the two different curves is which. Discuss any differences you see between the two plots and try to explain how those differences arise. How does this difference change as Δx gets smaller?

Answer: See Fig. 2. We can see that the numerical solution (solid blue line) under-estimates the exact solution ($y(x) = \tan{(x + C)}$). From Fig. 1, we can say that the difference between the exact and numerical solutions decreases as Δx gets smaller.
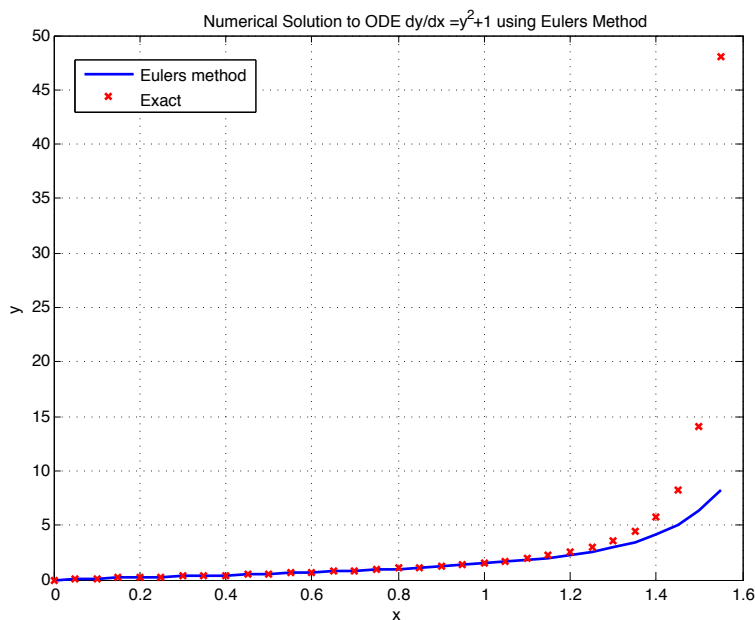
FIGURE 3. Comparison of an exact solution for the differential equation $y' = y^2 + 1$ with a solution with the same initial condition estimated numerically using Euler's method with a step-size of $\Delta x = 0.05$.

(4) Extend the range of comparison between the numerical solution and the exact solution to $x \in [0, \pi/2]$. How does your numerical solution compare to the analytical solution as x gets close to $\pi/2$? Explain why one is larger than the other.

Answer: From Fig. 3, we can see that the numerical solution gets worse and worse (as it under-estimates) the exact solution, which diverges as $x \to \pi/2$. The derivative of the exact solution also goes to ∞.

Remember that Euler's method assumes that the derivative is constant over the interval, using the value at the left-hand side. Since the derivative becomes increasingly larger as $x \to \pi/2$, it makes sense that the total error in the numerical approximation will increase and that the numerical solution will be an under-esitmate (since the assumed value of the derivative is always the smallest of anywhere on the actual interval).
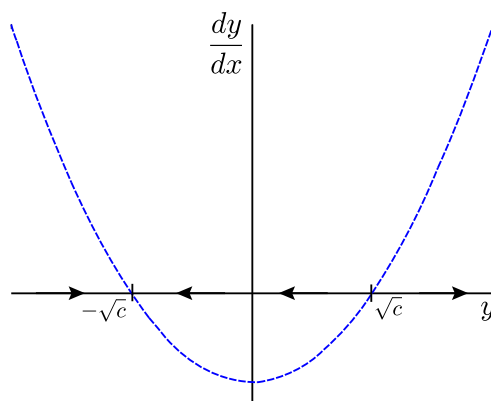
FIGURE 4. Phase-line portrait for $dy/dx = y^2 - c$. Note that the stability of the equilibria can be determined by whether the quadratic function is above or below the y-axis.

Part II

Consider the differential equation

$$\frac{dy}{dx} = y^2 - c$$

where c is a constant greater than zero.

(1) Determine all equilibrium solutions and their stability.

Answer: Factoring the right-hand side of the differential equation, we can see that there are two equilibria, at $y = \pm\sqrt{c}$. There are many ways one could determine their stability (e.g. use DFIELD, linearize about the fixed point and find the eigenvalues, etc..).

One approach is to plot the phase-line portrait (Fig.4). The right-hand side is just a concave-up parabola and the points where it crosses the horizontal-axis indicate the equilibrium locations. Whether the curve is above or below the axis indicates the directionality of the phase-line. We can see that $y = \sqrt{c}$ is unstable and $y = -\sqrt{c}$ is a stable equilibrium.

(2) Solve this equation analytically to obtain an expression for $y(x)$. Your answer should depend upon c and contain an arbitrary constant.

Answer: Similar to Part I, we would use separation of variables, then integrate both sides. For this case, we could use a partial fraction expansion to simplify the
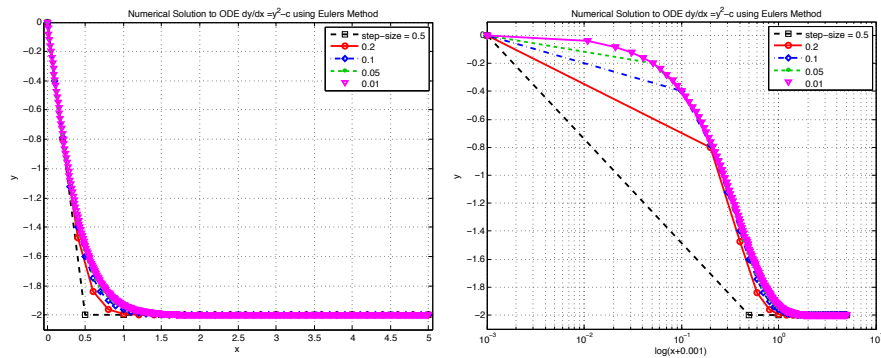
FIGURE 5. Comparison of an exact solution for the differential equation $y' = y^2 - c$ with solutions with the same initial condition estimated numerically using Euler's method for different step-sizes. The plot is also shown using a logarithmic x-axis (where the x values were slightly offset so to visualize the initial point).

denominator, yielding

$$\int \frac{dy}{y^2 - c} = \int \left[\frac{-1}{2\sqrt{c}} \frac{1}{(y + \sqrt{c})} + \frac{1}{2\sqrt{c}} \frac{1}{(y - \sqrt{c})} \right] \, dy = x + C$$

The resulting integral is easily solved (make a substitution and use $\int \frac{du}{u} = \ln|u| + c$) and solving for y, we have the final solution

$$y(x) = \sqrt{c} \, \frac{1 + Ae^{2\sqrt{c}x}}{1 - Ae^{2\sqrt{c}x}}$$

where A is an arbitrary constant.

(3) Write a code to solve the equation numerically using Euler's method on the interval $x \in [0, 5]$ for the initial condition $y(0) = 0$ and with $c = 4$. On a single figure, plot your estimated solution curve using the following step sizes for Δx: 0.5, 0.2, 0.1, 0.05, and 0.01. Make clear which curve corresponds to each step-size. How does the solution depend upon Δx?

Answer: An example code is given below as well as the figure (Fig. 5) that was asked for. The solution gives a better approximation for smaller values of Δx.

A sample code for solving the problem is shown below.

```
% ### odeSOLVEptII.m ### 11.10.08

% Matlab code to use Euler's method to solve the differential equation
% y' = y^2 - c
% (where c is a positive const.)

clear
```

```
clf

% ————————————-
% User Input Parameters

xMIN= 0;        % starting x-value
xMAX= 5;        % ending x-value
deltaX= 0.01;      % step-size

% initial conditions
y0= 0;

% ————————————-

x0= xMIN;
nsteps= (xMAX-xMIN)/deltaX;

yS(1)= y0; xS(1)= x0;

for nn=2:nsteps+1

    % note the difference here from the code for Part I
    yS(nn) = yS(nn-1) + deltaX*(yS(nn-1)^2 - c);

    % update x-array
    xS(nn) = xS(nn-1)+ deltaX;

end

% plot the numerical solution
plot(xS,yS)
hold on;
xlabel('x')
ylabel('y')
title('Numerical Solution to ODE dy/dx =y^2-c using Eulers Method')

% plot exact (i.e., analytically-derived) solution (only true if y(0)=0)
yA= sqrt(c)*(1-exp(2*sqrt(c)*xS))./((1+exp(2*sqrt(c)*xS)));
% could also have used -sqrt(c)*tanh(sqrt(c)*xS) here too

plot(xS,yA,'rx')
grid on
legend('Eulers method','Exact','Location','SouthEast')
```

(4) Using $\Delta x = 0.01$, find solution curves for different initial conditions $y(0) = y_o$. How do the solutions depend upon y_o?

Answer: See Fig.6.

(5) Explain your answer to the last part in terms of your analytic solution. Are the two results consistent?

Answer: The constant A will depend upon the initial condition such that

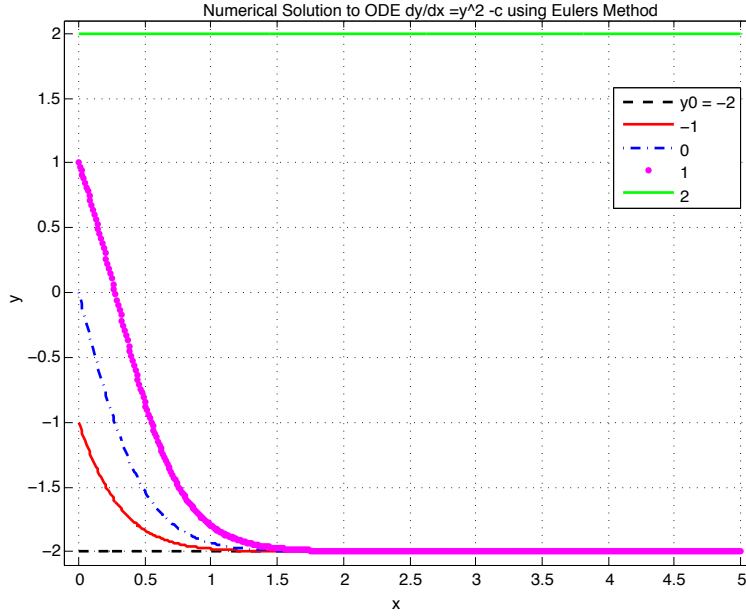$$A = \frac{y_o - \sqrt{c}}{y_o + \sqrt{c}}$$

FIGURE 6. Solutions for the differential equation $y' = y^2 - 4$ estimated numerically using Euler's method with a step-size of $\Delta x = 0.01$ for different initial y values at $x = 0$. Though not shown on this figure, initial conditions where $y_o > 2$ will diverge towards $+\infty$ (as $x \to \infty$) and those with $y_o < -2$ will converge towards $y = 2$.

where $y_o = y(0)$. Solutions will diverge towards $+\infty$ (for increasing x) if $y_o > \sqrt{c}$. Solutions will be sigmoidal when $|y_o| < \sqrt{c}$. The shape will be a reverse-S since solutions will move away from the unstable equilibrium (i.e., $\sqrt{c}$) and towards the stable one ($-\sqrt{c}$) as x increases. Decreasing y_o from around 2 towards -2 will move the center of the S-shape to the left. For $y_o < -\sqrt{c}$, solution curves will asymptotically approach the stable equilibrium forward in time, and diverge to $-\infty$ backwards in time. When $|y_o| = \sqrt{c}$, the solutions will be constant (i.e., equilibrium solutions). Both the numerical ($\Delta x = 0.01$) and analytical solutions are consistent with one another, though the numerical solution will underestimate for the case $y_o > \sqrt{c}$ as the solution diverges.

Note that for the initial condition $y(0) = 0$, we have the solution

$$y(x) = \sqrt{c}\, \frac{1 - e^{2\sqrt{c}x}}{1 + e^{2\sqrt{c}x}} = -\sqrt{c}\tanh\left(\sqrt{c}x\right)$$

.

(6) What is the effect of varying c? Explain in the contexts of both your analytical answer and numerical simulations. Do both agree?

Answer: The effect of changing c is two-fold. First, it changes the equilibrium values and thus what the asymptotic limits are. Second, because $\sqrt{c}$ appears in the argument of the exponent, increasing c will increase the rate at which solution asymptotically approach (or move away from) the equilibria. For example, if $|y_o| < \sqrt{c}$, then a larger value of c means a sharper transition in the S-shape that the solution takes.

One could also introduce the variables $u = y/\sqrt{c}$ and $\tau = \sqrt{c}x$. This change of variables would reformulate the differential equation as

$$\frac{du}{d\tau} = u^2 - 1$$

and thereby removing the parameter-dependence in terms of understanding the underlying dynamics.