Math 129 Section 005H
Lecture 30: Taylor series (§10.2)

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Note: Here is a PDF version of this file.

Reminders

- Mon 11/01: WA 10.1
- Tue 11/02: written HW10
- Wed 11/03: WA 10.2
- Fri 11/05: Exam 3
- Yellowdig
- Tutoring

Last time: Taylor polynomials

Given a function f, its *Taylor polynomial of order n centered at $x = 0$* is the polynomial

$$P_n(x) = C_0 + C_1 x + \cdots + C_n x^n$$

such that

$$\begin{aligned} P_n(0) &= f(0) \\ P_n''(0) &= f''(0) \\ &\vdots \\ P_n^{(n)}(0) &= f^{(n)}(0) \end{aligned}$$

We found

$$C_n = \frac{f^{(n)}(0)}{n!}$$

so

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

Similarly, the *Taylor polynomial of order n centered at $x = a$* is

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Warm-up. What is the nth Taylor polynomial of e^x about $x = 0$?

Taylor series

If we let $n \to \infty$ in the nth order Taylor polynomial of $f(x)$ about $x = 0$, we arrive at the *Taylor series*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

Similarly, there is a Taylor series about $x = a$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Note that when $n = 0$ and $x = a$, the above is

$$\frac{f(0) \cdot 0^0}{0!} + \frac{f'(0) \cdot 0^1}{1!} + + \frac{f''(0) \cdot 0^b}{2!} + \cdots$$

by convention, we define $0^0 = 1$ in this context.

Observe that Taylor polynomials are the partial sums of Taylor series.

Examples

We covered

- e^x about $x = 0$
- $\cos(x)$ about $x = 0$
- $\sin(x)$ about $x = 0$
- $\ln(x)$ about $x = 1$

Note on radius of convergence

One of you asked today whether the radius of convergence of the Taylor series depends on its center a. The answer is "it depends." One can show that if

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

for all real x, i.e., the radius of convergence R of the seris on the right is ∞, and $f(x)$ *equals* its Taylor series, then the radius of convergence is also ∞ for the Taylor series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

of f about any other point a.

This is not true for Taylor series with finite radius of convergence. For example, consider

$$f(x) = \frac{1}{x}.$$

Its Taylor series about $x = 1$ is easy to find without differentiating, by a little trickery:

$$\begin{aligned} \frac{1}{x} &= \frac{1}{(x-1)+1} \\ &= \frac{1}{a} \cdot \frac{1}{1+y} \end{aligned}$$

where $y = x - 1$. But the last expression is the sum of a geometricq series:

$$\begin{aligned} \frac{1}{1+y} &= 1 - y + y^2 - y^3 + \cdots \\ &= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots \end{aligned}$$

So

$$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots$$

We know this converges if and only if $|y| < 1$, or

$$|x1| < 1$$

so the radius of convergence of the Taylor series of $1/x$ about $x = 1$ is just $R = 1$.

Let's now do this about $x = 2$:

$$\begin{aligned} \frac{1}{x} &= \frac{1}{(x-2)+2} \\ &= \frac{1}{2} \cdot \frac{1}{1+(x-2)/2} \\ &= \frac{1}{2} \cdot \frac{1}{1+y} \end{aligned}$$

where now $y = (x-2)/2$. Again, the last expression is a geometric series:

$$\begin{aligned} \frac{1}{1+y} &= 1 - y + y^2 - y^3 + \cdots \\ &= 1 - \left(\frac{x-2}{2}\right) + \left(\frac{x-2}{2}\right)^2 - \left(\frac{x-2}{2}\right)^3 + \cdots \end{aligned}$$

So

$$\frac{1}{x} = \frac{1}{2}\left(1 - \left(\frac{x-2}{2}\right) + \left(\frac{x-2}{2}\right)^2 - \left(\frac{x-2}{2}\right)^3 + \cdots\right)$$

This converges if and only if $|y| < 1$, or

$$\left|\frac{x-2}{2}\right| < 1$$

which is equivalent to

$$|x-2| < 2.$$

So the radius of convergence is $R = 2$.

In general, the Taylor series of $1/x$ about $x = a$ has radius of convergence $R = |a|$. The reason for this is actually the vertical asymptote at $x = 0$: because the function itself diverges at $x = 0$, there is no way its Taylor series can converge. (What can it possibly converge to?) So even without the analysis above, we know that the radius of convergence of the Taylor series of $1/x$ about $x = a$ cannot possibly be larger than $|a|$.