Green's functions

Suppose that we want to solve a linear, inhomogeneous equation of the form

$$\mathcal{L}u(\mathbf{x}) = f(\mathbf{x}) \tag{1}$$

where u, f are functions whose domain is Ω. It happens that differential operators often have inverses that are integral operators. So for equation (1), we might expect a solution of the form

$$u(\mathbf{x}) = \int_\Omega G(\mathbf{x}, \mathbf{x}_0) f(\mathbf{x}_0) d\mathbf{x}_0. \tag{2}$$

If such a representation exists, the kernel of this integral operator $G(\mathbf{x}, \mathbf{x}_0)$ is called the Green's function.

It is useful to give a physical interpretation of (2). We think of $u(\mathbf{x})$ as the response at $\mathbf{x}$ to the influence given by a source function $f(\mathbf{x})$. For example, if the problem involved elasticity, u might be the displacement caused by an external force f. If this were an equation describing heat flow, u might be the temperature arising from a heat source described by f. The integral can be though of as the sum over influences created by point sources at each value of $\mathbf{x}_0$. But what is a point source? After all, there are no continuous functions which are nonzero just at one point. A new set of objects must be created to deal with this issue.

1 The delta function and distributions

Consider the ordinary differential equation

$$x'(t) + x(t) = f(t), \quad x(0) = 0. \tag{3}$$

We can easily solve this by multiplying by the integrating factor $\exp(t)$ and integrating to get

$$x(t) = \int_0^t \exp(s-t) f(s) ds. \tag{4}$$

Suppose $f(t)$ is chosen to represent a pulse of duration $\epsilon \ll 1$, occurring at time t_0. For $t < t_0$, $x(t) = 0$, and for $t > t_0 + \epsilon$,

$$x(t) = \int_{t_0}^{t_0+\epsilon} \exp(s-t) f(s) ds \approx \exp(t_0 - t) \int_{t_0}^{t_0+\epsilon} f(s) ds = I \exp(t_0 - t),$$

where I is the "impulse" - the integral of $f(t)$ over its support. Observe that the details of f do not matter, only what the integral is. In this respect, it would be nice to define a function with unit integral that is supported at just t_0.

There is a great need in differential equations (and elsewhere) to define objects that arise as limits of functions and behave like functions under integration but are not, properly speaking, functions themselves. These objects are sometimes called *generalized functions* or *distributions*. The most basic one of these is the so-called δ-function.

For each $\epsilon > 0$, define the family of ordinary functions

$$\delta_\epsilon(x) = \frac{1}{\epsilon\sqrt{\pi}} e^{-x^2/\epsilon^2}. \tag{5}$$

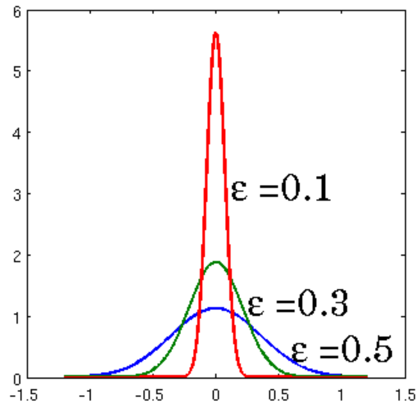

Figure 1: Approximations of the δ-function.

When ϵ is small, the graph of δ_ϵ (figure 1) is essentially just a spike at $x = 0$, but the integral of δ_ϵ is exactly one for any ϵ. For any continuous function $\phi(x)$, the integral of $\phi(x)\delta_\epsilon(x - x_0)$ is concentrated near the point x_0, and therefore

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\phi(x)\delta_\epsilon(x-x_0)dx = \lim_{\epsilon\to 0}\phi(x_0)\int_{-\infty}^{\infty}\delta_\epsilon(x-x_0)dx = \phi(x_0).$$

On the other hand, taking the limit $\epsilon \to 0$ *inside* the integral makes no sense: the limit of δ_ϵ is not a function at all! To get around this, we define a new object, $\delta(x - x_0)$, to behave as follows:

$$\int_{-\infty}^{\infty}\phi(x)\delta(x-x_0)dx = \phi(x_0). \tag{6}$$

Informally speaking, the δ-function "picks out" the value of a continuous function $\phi(x)$ at one point.

There are δ-functions for higher dimensions also. We define the n-dimensional δ-function to behave as

$$\int_{\mathbb{R}^n}\phi(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}_0)d\mathbf{x} = \phi(\mathbf{x}_0),$$

for any continuous $\phi(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$. Sometimes the multidimensional δ-function is written as a product of one dimensional ones: $\delta(\mathbf{x}) = \delta(x)\cdot\delta(y)\cdot\ldots$.

1.1 A more precise definition

To be concrete about distributions, one needs to talk about how they "act" on smooth functions. Note that (6) can be thought of as a linear mapping from smooth functions $\phi(x)$ to real numbers by the process of evaluating them at x_0. Linear mappings from a vector space (in this case, a space of smooth functions like ϕ) to the real numbers are often called *linear functionals*.

Now we come to the precise definition. A *distribution* is a continuous linear functional on the set of infinitely differentiable functions with bounded support; this space of functions is denoted by $\mathcal{D}$. We can write $d[\phi] : \mathcal{D} \to \mathbb{R}$ to represent such a map: for any input function ϕ, $d[\phi]$ gives us a number. Linearity means that

$$d[c_1\phi_1(x) + c_2\phi_2(x)] = c_1 d[\phi_1(x)] + c_2 d[\phi_2(x)].$$

Since integrals have properties similar to this, distributions are often defined in terms of them.

The class of distributions includes just plain old functions in the following sense. Suppose that $g(x)$ is integrable, but not necessarily continuous. We define the corresponding linear functional to be

$$g[\phi] = \int_{-\infty}^{\infty} g(x)\phi(x)dx.$$

(notice the subtlety of notation: $g(x)$ means the function evaluated at x, whereas the bracket $g[\phi]$ means multiply by function ϕ and integrate). Conversely, while all distributions cannot be associated with functions, they can be *approximated* by smooth, ordinary functions. This means that, for any distribution d, there exists a sequence of smooth functions $d_n(x) \in \mathcal{D}$ so that

$$d[\phi] = \lim_{n\to\infty} \int d_n(x)\phi(x)dx, \quad \text{for all } \phi \in \mathcal{D}.$$

For example, the sequence δ_ϵ that we first investigated comprises a set of smooth approximations to the δ-distribution.

We can now define what it means to integrate a distribution (notated as if it were a regular function $d(x)$) simply by setting

$$\int_{-\infty}^{\infty} d(x)\phi(x)dx \equiv d[\phi], \quad \text{for any } \phi \in \mathcal{D}.$$

This is consistent with the formula (6) since $\delta(x)$ maps a function ϕ onto its value at zero.

Here are a couple examples. A linear combination of two delta functions such as $d = 3\delta(x - 1) + 2\delta(x)$ defines a distribution. The corresponding linear functional is

$$d[\phi] = 3\phi(1) + 2\phi(0) = \int_{-\infty}^{\infty} d(x)\phi(x)dx.$$

The operation $d[\phi] = \phi'(0)$ takes a continuously differentiable function and returns the value of its derivative at zero. It is a linear mapping, since

$$d[c_1\phi_1 + c_2\phi_2] = c_1\phi_1'(0) + c_2\phi_2'(0) = c_1 d[\phi_1] + c_2 d[\phi_2].$$

A sequence of smooth functions that approximates this is $-\delta_\epsilon'(x)$, since integration by parts gives

$$\lim_{\epsilon\to 0} \int -\delta_\epsilon'(x)\phi(x)dx = \lim_{\epsilon\to 0} \int \delta_\epsilon(x)\phi'(x)dx = \phi'(0).$$

1.2 Distributions as derivatives

One useful aspect of distributions is that they make sense of derivatives of functions which are not differentiable. In fact, distributions themselves have derivatives that are distributions.

Suppose that $g(x)$ is an integrable function, but cannot be differentiated everywhere in its domain. It can make sense, however, to talk about *integrals* involving g'. Though integration by parts doesn't technically hold in the usual sense, for $\phi \in \mathcal{D}$ we can define

$$\int_{-\infty}^{\infty} g'(x)\phi(x)dx \equiv -\int_{-\infty}^{\infty} g(x)\phi'(x)dx.$$

Notice that the expression on the right makes perfect sense as a usual integral. We define the *distributional derivative* of $g(x)$ to be a distribution $g'[\phi]$ so that

$$g'[\phi] \equiv -g[\phi'].$$

More generally, for a distribution $d[\phi]$, we define its derivative as

$$d'[\phi] \equiv -d[\phi'].$$

Thus both regular non-differentiable functions and distributions may have derivatives which are distributions.

For example, if $H(x)$ is the step function

$$H(x) = \begin{cases} 0 & x < 0, \\ 1 & x \geq 0, \end{cases}$$

the distributional derivative of H is given by the rule

$$H'[\phi] = -H[\phi'] = -\int_{-\infty}^{\infty} H(x)\phi'(x)dx = -\int_{0}^{\infty} \phi'(x)dx = \phi(0).$$

Therefore, $\delta(x)$ is the the distributional derivative of the unit step function. Any function with a jump discontinuity will have a derivative that includes a scaled δ-function located at the discontinuity.

We can repeatedly differentiate a distribution as well. In this case, the n-th derivative is given as a distribution with rule

$$d^{(n)}[\phi] \equiv (-1)^n d[\phi^{(n)}].$$

For example, the second derivative of $f(x) = |x|$ can be regarded as a distribution

$$\begin{aligned}
(-1)^2 f[\phi''] &= \int_{-\infty}^{\infty} |x|\phi''(x)dx \\
&= -\int_{-\infty}^{0} x\phi''(x)dx + \int_{0}^{\infty} x\phi''(x)dx \\
&= -x\phi'(x)|_{-\infty}^{0} + x\phi'(x)|_{0}^{\infty} + \phi(x)|_{-\infty}^{0} - \phi(x)|_{0}^{\infty} = 2\phi(0).
\end{aligned}$$

So the second derivative of $|x|$ in the distributional sense is $2\delta(x)$.

Returning to problem (3), we now let $f(t) = \delta(t - t_0)$, and therefore the solution (4) is

$$x(t) = \int_{0}^{t} \exp(s - t)\delta(t - t_0)ds = \begin{cases} 0 & t < t_0, \\ \exp(t_0 - t) & t \geq t_0, \end{cases}$$

which is exactly what we expected. Notice that this solution is discontinuous, so it certainly can't solve the equation in the traditional sense. On the other hand, we can compute a distributional derivative:

$$x'[\phi] = -\int_{-\infty}^{\infty} x(t)\phi'(t)dt = -\int_{t_0}^{\infty} x(t)\phi'(t)dt = x(t_0)\phi(t_0) + \int_{t_0}^{\infty} x'(t)\phi(t)dt.$$

Notice the last integral can be written as

$$-\int_{-\infty}^{\infty} H(t - t_0)\exp(t_0 - t)\phi(t)dt,$$

therefore

$$x'(t) = \delta(t - t_0) - H(t - t_0)\exp(t_0 - t),$$

and $x'(t) + x(t) = \delta(t - t_0)$ as promised.

1.3 Relationship to Green's functions

Part of the problem with the definition (2) is that it doesn't tell us how to construct G. It is useful to imagine what happens when $f(\mathbf{x})$ is a point source, in other words $f(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}_i)$. Plugging into (2) we learn that the solution to

$$\mathcal{L}u(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}_i) \quad + \text{homogeneous boundary conditions} \tag{7}$$

should be

$$u(\mathbf{x}) = \int_\Omega G(\mathbf{x}, \mathbf{x}_0)\delta(\mathbf{x}_0 - \mathbf{x}_i)d\mathbf{x}_0 = G(\mathbf{x}, \mathbf{x}_i). \tag{8}$$

In other words, we find that the Green's function $G(\mathbf{x}, \mathbf{x}_0)$ formally satisfies

$$\mathcal{L}_x G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0) \tag{9}$$

(the subscript on $\mathcal{L}$ is needed since the linear operator acts on $\mathbf{x}$, not $\mathbf{x}_0$). This equation says that $G(\mathbf{x}, \mathbf{x}_0)$ is the influence felt at $\mathbf{x}$ due to a point source at $\mathbf{x}_0$.

Equation (9) is a more useful way of defining G since we can in many cases solve this "almost" homogeneous equation, either by direct integration or using Fourier techniques. In particular,

$$\mathcal{L}_x G(\mathbf{x}, \mathbf{x}_0) = 0, \quad \text{when } \mathbf{x} \neq \mathbf{x}_0, \tag{10}$$

which is a homogeneous equation with a "hole" in the domain at $\mathbf{x}_0$. To account for the δ-function, we can formally integrate both sides of $\mathcal{L}_x G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0)$ on any region containing $\mathbf{x}_0$. It is usually sufficient to allow these regions to be some ball $B_r(\mathbf{x}_0) = \{\mathbf{x} | |\mathbf{x} - \mathbf{x}_0| < r\}$, so that

$$\int_{B_r(\mathbf{x}_0)} \mathcal{L}_x G(\mathbf{x}, \mathbf{x}_0) dx = 1. \tag{11}$$

Equation (11) is called the *normalization condition*, and it is used to get the right "size" of the discontinuity or singularity of G at $\mathbf{x}_0$. In one dimension, this condition is more often called the *jump condition*.

In addition to (10-11), G must also satisfy the same type of homogeneous boundary conditions that the solution u does in the original problem. The reason for this is straightforward. Take, for example, the case of a homogeneous Dirichlet boundary condition $u = 0$ for $x \in \partial\Omega$. For any point x on the boundary, it must be the case that

$$\int_\Omega G(\mathbf{x}, \mathbf{x}_0) f(\mathbf{x}_0) d\mathbf{x}_0 = 0. \tag{12}$$

Since this must be true for any choice of f, it follows that $G(\mathbf{x}, \mathbf{x}_0) = 0$ for boundary points $\mathbf{x}$ (note that $\mathbf{x}_0$ is treated as a constant in this respect, and can be any point in the domain).

2 Green's functions in one dimensional problems

It is instructive to first work with ordinary differential equations of the form

$$\mathcal{L}u \equiv u^{(n)}(x) + F(u^{(n-1)}(x), u^{(n-2)}(x), \ldots) = f(x),$$

subject to some kind of boundary conditions, which we will initially suppose are homogeneous.

Following the previous discussion, the Green's function $G(x, x_0)$ satisfies (9), which is

$$G^{(n)} + F(G^{(n-1)}, G^{(n-2)}, \dots) = \delta(x - x_0), \tag{13}$$

where $G^{(n)} = \partial^n / \partial x^n$. This means that away from the point x_0

$$G^{(n)}(x) + F(G^{(n-1)}(x), G^{(n-2)}(x), \dots) = 0, \quad x > x_0 \tag{14}$$

$$G^{(n)}(x) + F(G^{(n-1)}(x), G^{(n-2)}(x), \dots) = 0, \quad x < x_0, \tag{15}$$

Note this represents *two* separate differential equations to be solved independently to begin with.

Formal integration of both sides of (13) gives

$$G^{(n-1)} = H(x - x_0) + \text{some continuous function}$$

This means that something special happens at x_0: the $n-1$-th derivative is not continuous, but suffers a discontinuous jump there. Integrating again shows that $G^{(n-2)}$ is continuous. These facts can be summarized as "connection" conditions at x_0:

$$\lim_{x \to x_0^+} \frac{\partial^{n-1} G}{\partial x^{n-1}} - \lim_{x \to x_0^-} \frac{\partial^{n-1} G}{\partial x^{n-1}} = 1, \quad \lim_{x \to x_0^+} \frac{\partial^m G}{\partial x^m} - \lim_{x \to x_0^-} \frac{\partial^m G}{\partial x^m} = 0, \quad m \le n - 2. \tag{16}$$

For first order equations, (16) means that G itself must have a jump discontinuity. For second order equations, G is continuous but its derivative has a jump discontinuity.

The problem for determining the Green's function is now very concrete, and simply uses elementary ODE techniques. First, (14) and (15) are solved separately. Then the general solution to (14) must be made to satisfy the right-hand boundary conditions only, whereas the solution to (15) must satisfy the left-hand boundary conditions. This will leave free constants in the piecewise solution for G, and these are uniquely determined by demanding that the connection conditions (16) are all met.

Example. Suppose $u : \mathbb{R} \to \mathbb{R}$ solves the ordinary differential equation and boundary conditions

$$u_{xx} = f(x), \quad u(0) = 0 = u(L). \tag{17}$$

The corresponding Green's function will solve

$$G_{xx}(x, x_0) = 0 \text{ for } x \neq x_0, \quad G(0, x_0) = 0 = G(L, x_0), \tag{18}$$

The connection conditions are

$$\lim_{x \to x_0^+} G_x(x, x_0) - \lim_{x \to x_0^-} G_x(x, x_0) = 1, \lim_{x \to x_0^+} G(x, x_0) = \lim_{x \to x_0^-} G(x, x_0). \tag{19}$$

Equation (18) actually represents *two* problems, one for $x < x_0$ and one for $x > x_0$. Their general solutions are found separately, and can be written

$$G(x, x_0) = \begin{cases} c_1 x + c_3 & x < x_0, \\ c_2(x - L) + c_4 & x > x_0. \end{cases} \tag{20}$$

Imposing the condition $G(0, x_0) = 0$ for the first and $G(L, x_0) = 0$ for the second gives

$$G(x, x_0) = \begin{cases} c_1 x & x < x_0, \\ c_2(x - L) & x > x_0. \end{cases} \tag{21}$$

Finally, using the connection conditions (19),

$$c_1 x_0 = c_2(x_0 - L), \quad c_2 - c_1 = 1,$$

so that $c_1 = (x_0 - L)/L$ and $c_2 = x_0/L$. The complete Green's function is therefore

$$G(x, x_0) = \begin{cases} (x_0 - L)x/L & x < x_0, \\ x_0(x - L)/L & x > x_0. \end{cases} \tag{22}$$

It follows that the solution to (17) can be written using G as

$$u(x) = \frac{1}{L}\left(\int_0^x x_0(x - L)f(x_0)dx_0 + \int_x^L x(x_0 - L)f(x_0)dx_0\right).$$

Notice the careful choice of integrands: the first integral involves the piece of the Green's function appropriate for $x_0 < x$, not the other way around.

Example. The one dimensional Helmholtz problem is

$$u_{xx} - k^2 u = f(x), \quad \lim_{x\to\pm\infty} u(x) = 0. \tag{23}$$

The corresponding Green's function therefore solves

$$G_{xx}(x, x_0) - k^2 G = 0 \text{ for } x \neq x_0, \tag{24}$$

together with $\lim_{x\to\pm\infty} G(x, x_0) = 0$ and connection conditions (19). The general solution of (24) is $G = c_1 \exp(-kx) + c_2 \exp(kx)$. For the piecewise Green's function to decay, the part for $x < x_0$ has $c_1 = 0$ and the part for $x > x_0$ has $c_2 = 0$, therefore

$$G(x, x_0) = \begin{cases} c_2 e^{kx} & x < x_0, \\ c_1 e^{-kx} & x > x_0. \end{cases} \tag{25}$$

Conditions (19) imply

$$c_2 \exp(kx_0) = c_1 \exp(-kx_0), \quad -kc_1 \exp(-kx_0) - kc_2 \exp(kx_0) = 1,$$

which yield $c_1 = -\exp(kx_0)/2k$ and $c_2 = -\exp(-kx_0)/2k$. The entire Green's function may then be written compactly as

$$G(x, x_0) = -\exp(-k|x - x_0|)/2k,$$

and the solution to (23) is

$$u(x) = -\frac{1}{2k}\int_{-\infty}^{\infty} f(x_0)e^{-k|x-x_0|}dx_0.$$

Notice that the Green's function in this problem simply depends on the distance between x and x_0; this is a very common feature in problems on infinite domains which have translation symmetry.

2.1 Using symmetry to construct new Green's functions from old ones

Green's functions depend both on a linear operator and boundary conditions. As a result, if the problem domain changes, a different Green's function must be found. A useful trick here is to use symmetry to construct a Green's function on a semi-infinite (half line) domain from a Green's function on the entire domain. This idea is often called the *method of images*.

Consider a modification of the previous example

$$\mathcal{L}u = u_{xx} - k^2 u = f(x), \quad u(0) = 0, \quad \lim_{x\to\infty} u(x) = 0. \tag{26}$$

We can't use the "free space" Green's function

$$G_\infty(x, x_0) = -\exp(-k|x - x_0|)/2k,$$

because it doesn't satisfy $G(0, x_0) = 0$ as required in this problem. Here's the needed insight: by subtracting G_∞ and its reflection about $x = 0$

$$G(x, x_0) = G_\infty(x, x_0) - G_\infty(-x, x_0)$$

does in fact satisfy $G(0, x_0) = 0$. On the other hand, does this proposed Green's function satisfy the right equation? Computing formally,

$$\mathcal{L}G(x, x_0) = \mathcal{L}G_\infty(x, x_0) - \mathcal{L}G_\infty(-x, x_0) = \delta(x - x_0) - \delta(-x - x_0).$$

The second delta function $\delta(-x - x_0)$ looks like trouble, but it is just zero on the interval $(0, \infty)$. Therefore $G_{xx} - k^2 G = \delta(x - x_0)$ when restricted to this domain, which is exactly what we want. The solution of (26) is therefore

$$u(x) = -\frac{1}{2k}\int_0^\infty f(x_0)\left[e^{-k|x-x_0|} - e^{-k|x+x_0|}\right] dx_0.$$

2.2 Dealing with inhomogeneous boundary conditions

Remarkably, a Green's function can be used for problems with inhomogeneous boundary conditions even though *the Green's function itself satisfies homogeneous boundary conditions*. This seems improbable at first since any combination or superposition of Green's functions would always still satisfy a homogeneous boundary condition. The way in which inhomogeneous boundary conditions enter relies on the so-called "Green's formula", which depends both on the linear operator in question as well as the *type* of boundary condition (i.e. Dirichlet, Neumann, or a combination).

Suppose we wanted to solve

$$u_{xx} = f, \quad u(0) = A, \quad u(L) = B, \tag{27}$$

using the Green's function $G(x, x_0)$ we found previously (22) when $A = 0 = B$. For this problem, the Green's formula is nothing more than integration by parts twice (essentially just the one dimensional Green's identity)

$$\int_0^L uv'' - vu''\, dx = [uv' - vu']_0^L. \tag{28}$$

To solve (27), we set $v(x) = G(x, x_0)$ in (28) and obtain

$$\int_0^L u(x)G_{xx}(x, x_0) - G(x, x_0)u''(x)dx = [u(x)G_x(x, x_0) - G(x, x_0)u'(x)]_{x=0}^{x=L}. \tag{29}$$

Using $u_{xx}(x) = f(x)$, $G_{xx}(x, x_0) = \delta(x - x_0)$ and noting that $G = 0$ if $x = 0$ or $x = L$, (29) collapses to

$$
\begin{aligned}
u(x_0) &= \int_0^L G(x, x_0) f(x) dx + [u(x) G_x(x, x_0)]_{x=0}^{x=L} \\
&= \int_0^L G(x, x_0) f(x) dx + B G_x(L, x_0) - A G_x(0, x_0).
\end{aligned} \tag{30}
$$

The first term on the right looks like the formula we had for homogeneous boundary conditions, with an important exception: x and x_0 are in the wrong places. We will resolve this apparent difference in section (3.5). The two terms at the end of this formula account for the inhomogeneous boundary conditions.

3 Green's functions in higher dimensions

3.1 The distributional Laplacian

In higher dimensions, one can make similar definitions of distributional derivatives by using Green's identities. For a twice differentiable function $u(\mathbf{x})$ and $\phi(\mathbf{x}) \in \mathcal{D}$, one has

$$
\int_{\mathbb{R}^n} (\Delta u) \phi \, dx = \int_{\mathbb{R}^n} u \, \Delta \phi \, dx,
$$

since $\phi(x)$ vanishes at infinity. This motivates a definition of the *distributional Laplacian* for functions $u(x)$ which are not entirely twice differentiable, which is a distribution with linear functional

$$
(\Delta u)[\phi] = u[\Delta \phi] = \int_{\mathbb{R}^n} u \, \Delta \phi \, dx. \tag{31}
$$

Example. Let's compute the distributional Laplacian of $f(\mathbf{x}) = 1/|\mathbf{x}|$, where $\mathbf{x} \in \mathbb{R}^3$. In spherical coordinates,

$$
\Delta(1/|\mathbf{x}|) = (\partial_{rr} + \frac{2}{r}\partial_r)\frac{1}{r} = 0,
$$

except when $|\mathbf{x}| = 0$. In order to capture the behavior at the origin, we need distributional derivatives instead. Using definition (31),

$$
\Delta f[\phi] = f[\Delta \phi] = \lim_{\epsilon \to 0} \int_{\mathbb{R}^3 / B_\epsilon(0)} \frac{\Delta \phi}{|\mathbf{x}|} \, dx,
$$

where $B_\epsilon(0)$ is a three dimensional ball of radius ϵ centered at the origin. The limit is needed because the integrand in unbounded, but for any $\epsilon > 0$, the Green's identity can be applied. Let $\partial/\partial n$ denote the normal derivative in the inward radial direction $\hat{\mathbf{n}} = -\mathbf{x}/|\mathbf{x}|$, so that

$$
\begin{aligned}
\Delta f[\phi] &= \lim_{\epsilon \to 0} \int_{\partial B_\epsilon(0)} \frac{1}{|\mathbf{x}|} \frac{\partial \phi}{\partial n} dx - \phi \frac{\partial}{\partial n} \left(\frac{1}{|\mathbf{x}|} \right) dx \\
&= \lim_{\epsilon \to 0} \left(\frac{1}{\epsilon} \int_{\partial B_\epsilon(0)} \frac{\partial \phi}{\partial n} - \frac{1}{\epsilon^2} \int_{\partial B_\epsilon(0)} \phi \, dx \right),
\end{aligned}
$$

where we used the fact that $1/|\mathbf{x}| = 1/\epsilon$ on the boundary $\partial B_\epsilon(0)$. Since $\partial B_\epsilon(0)$ is the surface of a sphere, we have

$$
\int_{\partial B_\epsilon(0)} \phi \, dx \sim 4\pi\epsilon^2 \phi(0), \quad \int_{\partial B_\epsilon(0)} \frac{\partial \phi}{\partial n} \, dx = \mathcal{O}(\epsilon^2),
$$

for small ϵ. The limit $\epsilon \to 0$ therefore yields

$$\Delta f[\phi] = -4\pi\phi(0).$$

We have discovered that $\Delta f = -4\pi\delta(\mathbf{x})$. The emergence of the delta function could not have been predicted without applying the definition!

3.2 Free space Green's functions

All of the ideas for one-dimensional Green's functions can be extended to higher dimensions. There are some extra complications that involve dealing with multidimensional derivatives and integrals, as well as coordinate systems (for example, note that a three dimensional Green's function depends on six scalar coordinates!). We are also more limited in finding Green's functions in the first place, since elementary ODE techniques do not always apply. We begin by deriving the most widely used Green's functions for the Laplacian operator on $\mathbb{R}^n$. Since there are no boundaries, these are called *free space* Green's functions.

Example: Laplacian in three dimensions Suppose $u : \mathbb{R}^3 \to \mathbb{R}$ solves

$$\Delta u = f, \qquad \lim_{|\mathbf{x}|\to\infty} u(\mathbf{x}) = 0. \tag{32}$$

In this case the homogeneous "boundary" condition is actually a far-field condition. The corresponding Green's function therefore must solve (10),

$$\Delta_x G(\mathbf{x}, \mathbf{x}_0) = 0, \quad \mathbf{x} \neq \mathbf{x}_0, \quad \lim_{|\mathbf{x}|\to\infty} G(\mathbf{x}, \mathbf{x}_0) = 0. \tag{33}$$

Using the divergence theorem, the normalization condition (11) is the same as (73)

$$\int_{\partial B_r(\mathbf{x}_0)} \frac{\partial_x G}{\partial n}(\mathbf{x}, \mathbf{x}_0)\, dx = 1. \tag{34}$$

for any ball B centered on x_0 and where the normal is directed outward. The notation ∂_x means that the normal derivative is with respect to $\mathbf{x}$ and not $\mathbf{x}_0$.

Let us observe the following: if we rotate the Green's function about $\mathbf{x}_0$, it still will solve (33-34) since the Laplace operator is invariant under rotation. Therefore G only depends on the distance between $\mathbf{x}$ and $\mathbf{x}_0$. Writing $G = g(r), r = |\mathbf{x} - \mathbf{x}_0|$, in spherical coordinates (33) is

$$\frac{1}{r^2}(r^2 g'(r))' = 0 \text{ if } r \neq 0, \quad \lim_{r\to\infty} g(r) = 0. \tag{35}$$

This is easily integrated twice to give the general solution

$$g(r) = -\frac{c_1}{r} + c_2, \tag{36}$$

where $c_2 = 0$ by using the far-field condition in (35). The normalization condition (34) determines c_1. Letting B be the unit ball centered at x_0 (whose surface area is 4π),

$$1 = \int_{\partial B} \frac{\partial_x G}{\partial n}(\mathbf{x}, \mathbf{x}_0)\, dx = \int_{\partial B} \frac{c_1}{r^2}\, dx = 4\pi c_1,$$

so that $c_1 = 1/4\pi$. Thus the Green's function is $G(\mathbf{x}, \mathbf{x}_0) = -1/(4\pi|\mathbf{x} - \mathbf{x}_0|)$, and the solution to (32) is

$$u(\mathbf{x}) = -\int_{\mathbb{R}^3} \frac{f(\mathbf{x}_0)}{4\pi|\mathbf{x} - \mathbf{x}_0|} dx_0.$$

Example: Laplacian in two dimensions. In this case we want to solve $\Delta u = f$ with $u, f : \mathbb{R}^2 \to \mathbb{R}$. Rather than stating a far-field side condition at this point, we first proceed as in the previous example. Looking for a Green's function of the form $G = g(|\mathbf{x} - \mathbf{x}_0|) \equiv g(r)$, in polar coordinates we find

$$\frac{1}{r}(rg'(r))' = 0 \text{ if } r \neq 0, \tag{37}$$

whose general solution is

$$g(r) = c_1 \ln r + c_2. \tag{38}$$

Notice that if we imposed $\lim_{|x|\to\infty} G(x, x_0) = 0$, then both c_1 and c_2 would need to be zero. An alternative is a "Neumann"-like condition $\lim_{|x|\to\infty} G_r = 0$, which would not determine c_2. In some applications, this can be successfully remedied by arbitrarily setting c_2 to 0. This can also be accomplished rigorously by using a (somewhat unusual looking) far-field condition for both $u(x)$ and $G(x, \cdot)$ of the form

$$\lim_{r\to\infty} \Big(u(r, \theta) - u_r(r, \theta) r \ln r \Big) = 0. \tag{39}$$

Using this condition for G in (38) gives

$$0 = \lim_{r\to\infty} c_1 \ln r + c_2 - (c_1/r) r \ln r = c_2,$$

so $c_2 = 0$. The normalization condition (34) gives

$$1 = \int_{\partial B} \frac{\partial_x G}{\partial n}(\mathbf{x}, \mathbf{x}_0)\, dx = \int_{\partial B} c_1\, dx = 2\pi c_1, \tag{40}$$

where B is the unit disk, so that $c_1 = 1/2\pi$. Thus the Green's function is $G(\mathbf{x}, \mathbf{x}_0) = \ln|\mathbf{x} - \mathbf{x}_0|/2\pi$, and the solution of the Poisson equation is

$$u(\mathbf{x}) = \int_{\mathbb{R}^2} \frac{\ln|\mathbf{x} - \mathbf{x}_0| f(\mathbf{x}_0)}{2\pi} dx_0^2.$$

It is sometimes useful to write G in polar coordinates. Using the law of cosines for the distance $|\mathbf{x} - \mathbf{x}_0|$, one gets

$$G(r, \theta; r_0, \theta_0) = \frac{1}{4\pi} \ln(r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0)) \tag{41}$$

(the semicolon is used here to visually separate sets of coordinates, by the way).

Example: The Helmholtz equation A two dimensional version of problem (23) is

$$\Delta u - u = f(\mathbf{x}), \quad \lim_{r\to\infty} u = 0, \quad u : \mathbb{R}^2 \to \mathbb{R}. \tag{42}$$

We can look for the Green's function for the Helmholtz operator $\mathcal{L} = \Delta - 1$ just as we did for the Laplacian, by supposing $G(\mathbf{x}, \mathbf{x}_0) = g(r)$ where $r = |\mathbf{x} - \mathbf{x}_0|$. Then since $\Delta G - G = 0$ when $\mathbf{x} \neq \mathbf{x}_0$, it follows that

$$g'' + \frac{1}{r} g' - g = 0, \quad \lim_{r\to\infty} g(r) = 0,$$

which is known as the modified Bessel equation of order zero. There is a single linearly independent solution which decays at infinity referred to as K_0, which happens to be represented by a nice formula

$$K_0(r) = \int_0^\infty \frac{\cos(rt)}{\sqrt{t^2+1}} dt.$$

The Green's function is therefore $G(\mathbf{x}, \mathbf{x}_0) = cK_0(|\mathbf{x} - \mathbf{x}_0|)$ where c is found from a normalization condition.

Applying the divergence theorem to (11) for $\mathcal{L} = \Delta - 1$ gives

$$1 = \int_{\partial B_r(\mathbf{x}_0)} \frac{\partial_x G}{\partial n}(\mathbf{x}, \mathbf{x}_0)\, dx - \int_{B_r(\mathbf{x}_0)} G(\mathbf{x}, \mathbf{x}_0) dx. \tag{43}$$

It can be shown (c.f. the venerable text of Bender & Orszag) that $K_0 \sim -\ln(r)$ when r is small, and therefore as $r \to 0$,

$$1 \sim -c \int_{\partial B_r(\mathbf{x}_0)} \frac{1}{r}\, dx = -2\pi c. \tag{44}$$

Using (43), it follows that $c = -1/2\pi$. The solution to (42) is therefore

$$u(\mathbf{x}) = -\int_{\mathbb{R}^2} \frac{K_0(|\mathbf{x} - \mathbf{x}_0|) f(\mathbf{x}_0)}{2\pi} dx_0.$$

3.3 Dealing with boundaries and the method of images

The examples in the previous section are *free space* Green's functions, since there are no domain boundaries. Recall that the Green's function must satisfy all the same homogeneous boundary conditions as underlying linear problem. Free space Green's functions typically satisfy conditions at infinity, but not along true boundaries. With a little cleverness, however, we can still employ them as a starting point to find Green's functions for certain bounded or semi-infinite domains.

3.3.1 Arbitrary bounded domains

Suppose that we wish to solve the Poisson equation for $u : \Omega \to \mathbb{R}$

$$\Delta u = f(x, y), \quad u = 0 \text{ on } \partial\Omega \tag{45}$$

where Ω is a bounded, open set in $\mathbb{R}^2$. We need a Green's function $G(\mathbf{x}, \mathbf{x}_0)$ which satisfies

$$\Delta_x G = \delta(\mathbf{x} - \mathbf{x}_0), \quad G(\mathbf{x}, \mathbf{x}_0) = 0 \text{ when } \mathbf{x} \in \partial\Omega. \tag{46}$$

It's tempting to use the free space Green's function $G_2(\mathbf{x}, \mathbf{x}_0)$
$= \ln|\mathbf{x} - \mathbf{x}_0|/2\pi$, which does indeed satisfy the equation in (46), but G_2 is not zero on the boundary $\partial\Omega$.

We should be thinking of (46) as an inhomogeneous equation, and use the method of particular solutions. In fact, G_2 is a particular solution, so if we write $G(\mathbf{x}, \mathbf{x}_0) = G_2(\mathbf{x}, \mathbf{x}_0) + G_R(\mathbf{x}, \mathbf{x}_0)$, then for each $\mathbf{x}_0 \in \Omega$ G_R solves a homogeneous equation with nonzero boundary data

$$\Delta_x G_R = 0, \quad G_R(\mathbf{x}, \mathbf{x}_0) = -G_2(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{2\pi} \ln|\mathbf{x} - \mathbf{x}_0| \text{ for } x \in \partial\Omega. \tag{47}$$

The function G_R is called the *regular part* of the Green's function, and it just a nice, usual solution of Laplace's equation, without any singular behavior. In other words, the desired Green's function has exactly the same logarithmic singularity as the free space version, but is quantitatively different far away from $\mathbf{x}_0$. In general, this method is used in conjunction with other techniques as a way to characterize Green's functions without actually solving for them.

3.3.2 Method of images

In section (2.1), we found that symmetry of domains can be exploited in constructing Green's functions using their free space counterparts. The basic inspiration is a simple observation about even and odd functions: for continuously differentiable $f(x) : \mathbb{R} \to \mathbb{R}$,

$$g(x) = f(x) - f(-x) \text{ is an odd function and } g(0) = 0, \tag{48}$$

$$h(x) = f(x) + f(-x) \text{ is an even function and } h'(0) = 0. \tag{49}$$

Therefore subtracting mirror images of a function will satisfy a Dirichlet-type boundary condition at $x = 0$, whereas their sum satisfies a Neumann-type boundary condition.

Example. Consider finding the Green's function for the upper-half space problem for $u : \mathbb{R}^3 \cap \{z > 0\} \to \mathbb{R}$:

$$\Delta u = f, \quad \lim_{|\mathbf{x}| \to \infty} u(\mathbf{x}) = 0, \quad u(x, y, 0) = 0. \tag{50}$$

The free space Green's function (in Cartesian coordinates) is

$$G_3(x, y, z; x_0, y_0, z_0) = -\frac{1}{4\pi\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}}$$

which is certainly not zero on the xy-plane. On the other hand, by virtue of (48), subtracting G from its mirror image about $z = 0$ should do the trick:

$$G(x, y, z; x_0, y_0, z_0) = G_3(x, y, z; x_0, y_0, z_0) - G_3(x, y, -z; x_0, y_0, z_0)$$
$$= \frac{1}{4\pi}\left(\frac{1}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}} - \frac{1}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z + z_0)^2}}\right).$$

Let's check that this is what we want. It is easy to see that $\lim_{|\mathbf{x}| \to \infty} G = 0$ and also that $G(x, y, 0; x_0, y_0, z_0) = 0$. The equation formally satisfied by G is

$$\Delta_x G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0) - \delta(\mathbf{x} - \mathbf{x}_0^*), \tag{51}$$

where $\mathbf{x}_0^* = (x_0, y_0, -z_0)$ is called the *image source*. The presence of the extra δ-function on the right hand side in (51) is not a problem, because a differential equation only needs to be satisfied in the domain where it is posed. Therefore

$$\Delta_x G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0), \quad \text{for all } \mathbf{x}, \mathbf{x}_0 \in \mathbb{R}^3 \cap \{z > 0\}.$$

Example: Green's function for a disk. A more clever use of the method of images is where Ω is a disk of radius a, and we want (in polar coordinates) $G(r, \theta; r_0, \theta_0)$ to solve $\Delta G = \delta(\mathbf{x} - \mathbf{x}_0)$ with boundary condition $G(a, \theta; r_0, \theta_0) = 0$. The idea is to subtract the two-dimensional free space Green's function G_2 from some carefully chosen "reflection" across the boundary of the disk, plus some constant. This results (after some guesswork, see below) in

$$G(r, \theta; r_0, \theta_0) = G_2(\mathbf{x}, \mathbf{x}_0) - G_2(\mathbf{x}, \mathbf{x}_0^*) + \frac{1}{2\pi}\ln(a/r_0) \tag{52}$$

$$= \frac{1}{4\pi}\ln\left(\frac{a^2}{r_0^2}\frac{r^2 + r_0^2 - 2rr_0\cos(\theta - \theta_0)}{r^2 + a^4/r_0^2 - 2ra^2/r_0\cos(\theta - \theta_0)}\right). \tag{53}$$

where the image source is outside the disk at $\mathbf{x}_0^* = a^2\mathbf{x}_0/r_0^2$. Does this work? First, we have $\Delta G = \delta(\mathbf{x}-\mathbf{x}_0) - \delta(\mathbf{x}-\mathbf{x}_0^*)$ as in the previous example, which is just $\Delta G = \delta(\mathbf{x}-\mathbf{x}_0)$ when restricted to the disk. Evaluating G on the boundary,

$$\begin{aligned} G(a,\theta;r_0,\theta_0) &= \frac{1}{4\pi}\ln\left(\frac{a^2}{r_0^2}\frac{a^2+r_0^2-2ar_0\cos(\theta-\theta_0)}{a^2+a^4/r_0^2-2a^3/r_0\cos(\theta-\theta_0)}\right) \\ &= \frac{1}{4\pi}\ln\left(\frac{a^2+r_0^2-2ar_0\cos(\theta-\theta_0)}{r_0^2+a^2-2ar_0\cos(\theta-\theta_0)}\right) = 0, \end{aligned}$$

which means G has the correct boundary condition as well.

So how did we find x_0^*? It is too much to ask that the difference of Greens functions $G_2(\mathbf{x},\mathbf{x}_0) - G_2(\mathbf{x},\mathbf{x}_0^*)$ is zero on the boundary $|x| = a$, but perhaps instead the difference is a constant, i.e.

$$\frac{1}{2\pi}\left(\ln|x-x_0| - \ln|x-x_0^*|\right) = C.$$

After exponentiation this is equivalent to

$$|\mathbf{x}-\mathbf{x}_0|^2 = k|\mathbf{x}-\mathbf{x}_0^*|^2, \quad k = e^{4\pi C}, \quad \text{when } |x| = a. \tag{54}$$

It is reasonable from symmetry considerations that $\mathbf{x}_0, \mathbf{x}_0^*$ are on the same line, in other words $\mathbf{x}_0^* = \gamma\mathbf{x}_0$. The law of cosines applied to (54) gives

$$a^2 + r_0^2 - 2ar_0\cos\phi = k(a^2 + \gamma^2 r_0^2 - 2\gamma a r_0\cos\phi), \quad \phi = \theta - \theta_0.$$

This must be true for all ϕ, so that

$$a^2 + r_0^2 = k(a^2 + \gamma^2 r_0^2), \quad 2ar_0 = 2ak\gamma r_0,$$

which leads to $\gamma = a^2/r_0^2$ and $k = 1/\gamma$, consistent with (52).

3.4 Inhomogeneous boundary conditions and the Green's formula representation

If a differential operator $\mathcal{L}$ is self-adjoint with respect to the usual L^2 inner product, then for all functions u, v satisfying the homogeneous boundary conditions in problem (1),

$$\int_\Omega (\mathcal{L}v)u\,d\mathbf{x} - \int_\Omega (\mathcal{L}u)v\,d\mathbf{x} = 0. \tag{55}$$

What if u, v don't necessarily satisfy homogeneous boundary conditions? Then something like (55) would still be true, but terms involving boundary values of u, v would appear:

$$\int_\Omega (\mathcal{L}v)u\,d\mathbf{x} - \int_\Omega (\mathcal{L}u)v\,d\mathbf{x} = \text{boundary terms involving } u \text{ and } v. \tag{56}$$

What this formula actually looks like depends on the linear operator in question and is known as the *Green's formula* for the linear operator $\mathcal{L}$. For the Laplacian, the associated Green's formula is nothing more than Green's identity, which reads

$$\int_\Omega u\Delta v - v\Delta u\,d\mathbf{x} = \int_{\partial\Omega} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\,d\mathbf{x}. \tag{57}$$

Suppose we wish to solve the problem with the inhomogeneous boundary condition

$$\Delta u = f \text{ in } \Omega, \quad u(\mathbf{x}) = h(\mathbf{x}) \ \mathbf{x} \in \partial\Omega.$$

Let G be the Green's function that solves $\Delta G = \delta(\mathbf{x} - \mathbf{x}_0)$ with homogeneous, Dirichlet boundary conditions. Substituting $v(\mathbf{x}) = G(\mathbf{x}, \mathbf{x}_0)$ in (57), we have (being careful to keep $\mathbf{x}$ as the variable of integration)

$$\int_\Omega u(\mathbf{x})\delta(\mathbf{x} - \mathbf{x}_0) - G(\mathbf{x}, \mathbf{x}_0)f(\mathbf{x})\, d\mathbf{x} = \int_{\partial\Omega} u(\mathbf{x})\frac{\partial_x G}{\partial n}(\mathbf{x}, \mathbf{x}_0) - G(\mathbf{x}, \mathbf{x}_0)\frac{\partial u}{\partial n}\, d\mathbf{x}. \tag{58}$$

Using the definition of the δ-function and the fact that G is zero on the domain boundary, this simplifies to

$$u(\mathbf{x}_0) = \int_\Omega G(\mathbf{x}, \mathbf{x}_0)f(\mathbf{x})d\mathbf{x} + \int_{\partial\Omega} h(\mathbf{x})\frac{\partial_x G}{\partial n}(\mathbf{x}, \mathbf{x}_0)\, d\mathbf{x}. \tag{59}$$

The first term is just the solution we expect for homogeneous boundary conditions. The second term is more surprising: it is a *derivative* of G that goes into the formula to account for the inhomogeneous Dirichlet boundary condition.

Example: the Poisson integral formula revisited. In the case that Ω is a disk of radius a, we have found the Green's function which is zero on the boundary in equation (52). The boundary value problem

$$\Delta u = 0, \quad u(a, \theta) = h(\theta)$$

has a solution given by (59), where $f(\mathbf{x}) = 0$. The normal derivative to the boundary of G is just the radial derivative

$$\begin{aligned}\frac{\partial_x G}{\partial n}(\mathbf{x}, \mathbf{x}_0) &= G_r(r, \theta; r_0, \theta_0)\\ &= \frac{1}{4\pi}\left(\frac{2r - 2r_0\cos(\theta - \theta_0)}{r^2 + r_0^2 - 2rr_0\cos(\theta - \theta_0)} - \frac{2rr_0^2 - 2r_0a^2\cos(\theta - \theta_0)}{r^2r_0^2 + a^4 - 2rr_0a^2\cos(\theta - \theta_0)}\right),\end{aligned}$$

which at $r = a$ is

$$\frac{a}{2\pi}\left(\frac{1 - (r_0/a)^2}{r_0^2 + a^2 - 2ar_0\cos(\theta - \theta_0)}\right).$$

The integral on the boundary can be parameterized using θ, which produces an extra factor of a from the arclength differential $|dx| = ad\theta_0$. Formula (59) becomes

$$u(r_0, \theta_0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(a^2 - r_0^2)h(\theta)}{a^2 + r_0^2 - 2ar_0\cos(\theta - \theta_0)}d\theta.$$

We obtained this result already using separation of variables; it is the Poisson integral formula.

Example: Neumann type boundary conditions. Suppose we want to solve Laplace's equation in the upper half space $\{(x, y, z)|z > 0\}$, with both a far-field boundary condition and a Neumann condition on the xy-plane:

$$\Delta u = 0, \quad \lim_{z\to\infty} u(x, y, z) = 0, \quad u_z(x, y, 0) = h(x, y). \tag{60}$$

Notice that the Green's formula (57) has boundary terms than involve both Dirichlet and Neumann data. Of course, we only know the derivative of u on the boundary, so we need to make sure that boundary terms involving Dirichlet data will vanish. To make this happen, the Green's function substituted in for v must have $\partial_x G/\partial n = 0$ on the boundary. In other words, we must respect the "boundary condition principle":

> The Green's function must have the same type of boundary conditions
> as the problem to be solved, and they must be homogeneous.

For (60), we need a Green's function which has $G_z(x, y, 0; x_0, y_0, z_0) = 0$, and vanishes at infinity. The method of images tells us this can be done using the *even* reflection across the xy-plane. In other words, we want

$$\begin{aligned} G(x,y,z;x_0,y_0,z_0) &= G_3(x,y,z;x_0,y_0,z_0) + G_3(x,y,z;x_0,y_0,-z_0) \\ &= \frac{1}{4\pi}\left(\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} + \frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z+z_0)^2}}\right). \end{aligned}$$

It is easy to check $G_z = 0$ when $z = 0$. Now substituting $v(\mathbf{x}) = G(\mathbf{x}, \mathbf{x}_0)$ into (57), we obtain after collapsing the δ-function integral

$$u(\mathbf{x}_0) = -\int_{\partial\Omega} G(\mathbf{x},\mathbf{x}_0)\frac{\partial u}{\partial n}(\mathbf{x})\, d\mathbf{x}, \tag{61}$$

since $\Delta u = 0$ and $\partial G/\partial z$ vanishes both at infinity and at $z = 0$. Notice that since $\hat{\mathbf{n}}$ is directed *outward* relative to Ω, $\partial u/\partial n = -u_z(x, y, 0)$, so that in coordinates (61) becomes

$$u(x_0,y_0,z_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{h(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2+z_0^2}}dxdy. \tag{62}$$

This is another kind of Poisson's formula suitable for the upper half space.

3.5 Symmetry (reciprocity) of the Green's function

Occasionally we need to rearrange the arguments of a Green's function for self-adjoint operators. It turns out that the associated integral operator is self-adjoint as well, which means that $G(\mathbf{x}, \mathbf{x}_0) = G(\mathbf{x}_0, \mathbf{x})$. To demonstrate this fact, use the adjoint identity (55) with $v(\mathbf{x}) = G(\mathbf{x}, \mathbf{x}_1)$ and $u(\mathbf{x}) = G(\mathbf{x}, \mathbf{x}_2)$. Because of (7), we have $\mathcal{L}v = \delta(\mathbf{x} - \mathbf{x}_1)$ and $\mathcal{L}u = \delta(\mathbf{x} - \mathbf{x}_2)$. Plugging these into (55) gives

$$\int_\Omega \delta(\mathbf{x}-\mathbf{x}_1)G(\mathbf{x},\mathbf{x}_2)d\mathbf{x} - \int_\Omega G(\mathbf{x},\mathbf{x}_1)\delta(\mathbf{x}-\mathbf{x}_2)d\mathbf{x} = 0. \tag{63}$$

Using the basic property of the δ-function, this simplifies to

$$G(\mathbf{x}_1,\mathbf{x}_2) - G(\mathbf{x}_2,\mathbf{x}_1) = 0. \tag{64}$$

which is what we want to show. A related fact has to do with interchanging partial derivatives of G. Take the one-dimensional case, and observe

$$\begin{aligned} \partial_x G(x,x_0) &= \lim_{h\to 0}\frac{G(x+h,x_0)-G(x,x_0)}{h} \\ &= \lim_{h\to 0}\frac{G(x_0,x+h)-G(x_0,x)}{h} = \partial_{x_0}G(x_0,x). \end{aligned} \tag{65}$$

In other words, reciprocity implies that we can interchange the partial derivatives with respect to x and x_0 provided we also interchange the arguments x and x_0.

We can use these identities to rearrange our solution formulas. Take equation (30) for example. Using reciprocity for both $G()$ and G_x gives

$$u(x_0) = \int_0^L G(x_0, x) f(x) dx + [u(x) G_{x_0}(x_0, x)]_{x=0}^{x=L}. \tag{66}$$

The first term still does not look the same as the original idea (2), but we can play another trick: simply swap notation between $\mathbf{x}$ and $\mathbf{x}_0$, giving

$$u(x) = \int_0^L G(x, x_0) f(x_0) dx_0 + [u(x_0) G_{x_0}(x, x_0)]_{x_0=0}^{x_0=L} \tag{67}$$

(notice the partial derivative of G is unchanged; this requires a little thinking about the definition of a partial derivative!). A similar rearrangement of the formula (59) gives

$$u(\mathbf{x}) = \int_\Omega G(\mathbf{x}, \mathbf{x}_0) f(\mathbf{x}_0) d\mathbf{x}_0 + \int_{\partial\Omega} h(\mathbf{x}_0) \frac{\partial_{x_0} G}{\partial n}(\mathbf{x}, \mathbf{x}_0)\, d\mathbf{x}_0. \tag{68}$$

3.6 Distributions revisited

We conclude by showing more precisely that equation (10) and normalization condition (34) arise from the definition of distributional derivative for $\Delta_x G = \delta(\mathbf{x} - \mathbf{x}_0)$. The distributional Laplacian of G must be

$$\phi(\mathbf{x}_0) = \Delta G[\phi] = \int_{\mathbb{R}^n} G(\mathbf{x}, \mathbf{x}_0) \Delta\phi(x)\, dx \tag{69}$$

for every $\phi \in \mathcal{D}$. Let ϕ be any function whose support excludes $\mathbf{x}_0$. Assuming G is twice differentiable if $\mathbf{x} \neq \mathbf{x}_0$, Green's identity implies

$$\int_B \phi \Delta G(\mathbf{x}, \mathbf{x}_0)\, dx = \int_B G(\mathbf{x}, \mathbf{x}_0) \Delta\phi\, dx = \phi(\mathbf{x}_0) = 0, \tag{70}$$

by using (69). Since this is true for *any* choice of ϕ with $\phi = 0$ outside B, this means that

$$\Delta G(\mathbf{x}, \mathbf{x}_0) = 0, \quad \mathbf{x} \neq \mathbf{x}_0. \tag{71}$$

Conversely, take some $\phi \in \mathcal{D}$ with $\phi = 1$ on a ball B centered at $\mathbf{x}_0$. Using (69),

$$1 = \phi(0) = \int_{\mathbb{R}^n} G(\mathbf{x}, \mathbf{x}_0) \Delta\phi\, dx = \int_{\mathbb{R}^n/B} G(\mathbf{x}, \mathbf{x}_0) \Delta\, \phi dx, \tag{72}$$

since $\Delta\phi) = 0$ on B. The Green's identity can now be used on the remaining integral. by virtue of (71), and the fact that $\nabla\phi = 0$ on ∂B, we are left with

$$\int_{\partial B} \frac{\partial_x G(\mathbf{x}, \mathbf{x}_0)}{\partial n}\, dx = 1. \tag{73}$$

which is the same as (11) by virtue of the divergence theorem.