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Therefore u is a constant on lines parallel to \mathbf{v} , having form $x - ct = x_0$. These are known as *characteristic curves*.

It follows u is function of $x - ct$ alone:

$$u(x, t) = f(x - ct).$$

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More generally

$$au_x + bu_y = 0$$

has characteristic curves of the form $bx - ay = C$, and general solution is therefore

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Function $f()$ is determined by boundary conditions.

Remark: since $u(x, y)$ is constant along characteristic curves, boundary data must be compatible.

First order wave equations, example

Solve $u_x = u_y$ subject to $u(x, y) = \sin(x)$ on the line $y = x$.

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Complete solution is therefore $u(x, y) = \sin((x + y)/2)$.

Second order wave equations

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Can show particular solution has form $u_p = f(x + ct)$;
complimentary solution has form $u_{\text{hom}} = g(x - ct)$, therefore

$$u = u_{\text{hom}} + u_p = g(x - ct) + f(x + ct).$$

Second order wave equations,cont.

Now satisfy the initial conditions:

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = v_0(x)$$

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Inserting into general solution

$$\begin{aligned} f(x) + g(x) &= u_0(x), \\ f'(x) - g'(x) &= \frac{1}{c} v_0(x). \end{aligned}$$

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$$f(x) - g(x) = \frac{1}{c} \int_0^x v_0(x') dx' + K.$$

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Use method of elimination

$$\begin{aligned} f(x) &= \frac{1}{2} \left(u_0(x) + \frac{1}{c} \int_0^x v_0(x') dx' + K \right), \\ g(x) &= \frac{1}{2} \left(u_0(x) - \frac{1}{c} \int_0^x v_0(x') dx' - K \right) \end{aligned}$$

Second order wave equations, cont.

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Complete solution is therefore d'Alembert's formula

$$u = f(x+ct) + g(x-ct) = \frac{1}{2} (u_0(x+ct) + u_0(x-ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} v_0(x') dx'.$$

Generalized second order equation

Notice that

$$u_{xx} + (a - b)u_{xy} - abu_{yy} = 0$$

can be factored as

$$\left(\frac{\partial}{\partial x} + a \frac{\partial}{\partial y} \right) \left(\frac{\partial}{\partial x} - b \frac{\partial}{\partial y} \right) u = 0.$$

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Using same process as before, general solution is

$$u(x, y) = f(y - ax) + g(y + bx)$$

Generalized second order equation, example

Want a d'Alembert-type solution for $u_{xx} + u_{xy} - 20u_{yy} = 0$
subject to initial conditions $u(x, 0) = \phi(x)$ and $u_y(x, 0) = \psi(x)$.

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Factoring gives

$$\left(\frac{\partial}{\partial x} + 5 \frac{\partial}{\partial y} \right) \left(\frac{\partial}{\partial x} - 4 \frac{\partial}{\partial y} \right) u = 0.$$

so that general solution is

$$u(x, y) = g(4x + y) + f(5x - y).$$

Generalized second order equation, example

With general solution $u(x, y) = g(4x + y) + f(5x - y)$, satisfying initial data gives

$$\begin{aligned}g(4x) + f(5x) &= \phi(x), \\g'(4x) - f'(5x) &= \psi(x).\end{aligned}$$

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Using elimination to solve,

$$\begin{aligned}g(4x) &= \frac{4}{9}\phi(x) + \frac{20}{9} \int_0^x \psi(x') dx' + \frac{20C}{9}, \\f(5x) &= \frac{5}{9}\phi(x) - \frac{20}{9} \int_0^x \psi(x') dx' - \frac{20C}{9}.\end{aligned}$$

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It follows that

$$\begin{aligned}u(x, y) &= g(4x + y) + f(5x - y) = g(4(x + y/4)) + f(5(x - y/5)) \\&= \frac{4}{9}\phi(x + y/4) + \frac{4}{9}\phi(x - y/5) + \frac{20}{9} \int_{x-y/5}^{x+y/4} \psi(x') dx' .\end{aligned}$$

For d'Alembert's formula

$$u(x, t) = \frac{1}{2} (u_0(x + ct) + u_0(x - ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} v_0(x') dx'.$$

the solution at (x, t) only depends on the initial condition on $I_d(x, t) = (x - ct, x + ct)$. This interval is called the *domain of dependence*.

Wave speeds and domain of dependence

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Contrast to diffusion equation:

$$u(x, t) = \int_{-\infty}^{\infty} \frac{u(x_0, 0)}{\sqrt{4\pi Dt}} e^{-(x-x_0)^2/(4Dt)} dx_0.$$

where $I_d = \mathbb{R}$.