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or

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Plugging in gives dispersion relation  $\omega = \omega(k)$  or  $\sigma = \sigma(k)$ .

## Examples

For usual wave equation

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For the diffusion equation

$$u_t = Du_{xx},$$

same process gives  $\sigma(k) = -Dk^2$ , i.e. solutions decay of  $k \neq \text{zero}$ .

## Phase and group velocity of waves

For a real dispersion relation  $\omega(k)$ , there are solutions

$$u(x, t) = \exp\left(ikx - i\omega(k)t\right) = \exp\left(ik\left[x - \frac{\omega(k)}{k}t\right]\right),$$

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Integral is a traveling wave moving at speed  $\omega'(k_0)$ . This is known as the *group velocity*.

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**Animation of phase and group velocity**

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For  $B < 0$ ,  $\text{Re } \sigma < 0$ , therefore linearly stable.

For  $B > 0$ ,  $\text{Re } \sigma > 0$  for small  $k$ , therefore linearly unstable.

For  $B = 0$ , marginally stable since  $\text{Re } \sigma(0) = 0$ .

Consider generic linear or nonlinear PDE of form

$$u_t = R(u, u_x, u_{xx}, \dots)$$

## Steady state solutions

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A *steady state solution*  $u_0(x)$  has  $\partial u_0 / \partial t = 0$ ; it therefore solves

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Remarks:

- $u_0$  solves an ODE
- $u_0$  is usually subject to boundary/ far field conditions
- If  $u(x, 0) = u_0(x)$ , then  $u(x, t) = u_0(x)$  for all  $t > 0$ .
- Can be many solutions, esp. for nonlinear equations

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Solution is easy:  $u_0 = x$ .

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## Steady state solutions, example 2

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Trick to solving: multiply by  $u_x$  and integrate.

$$\int u_{xx} u_x + 2u(1 - u^2) u_x dx = \frac{1}{2} u_x^2 + u^2 - \frac{1}{2} u^4 + C = 0,$$

which uses  $u_{xx} u_x = \frac{1}{2} (u_x^2)_x$  and  $f'(u) u_x = f(u)_x$ .

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Since  $u(\pm\infty) = \pm 1$ ,  $C = -1/2$ .

## Steady state solutions, example 3, cont.

First order equation can now be written

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which can be solved by separating variables

$$\frac{du}{1 - u^2} = dx, \quad \text{therefore} \quad \frac{1}{2} \ln \left| \frac{1 + u}{1 - u} \right| = x + c$$

so that

$$u(x) = \tanh(x + c).$$

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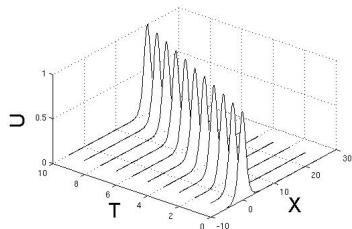
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$$\frac{1}{2}u_x^2 - \frac{V}{2}u^2 + u^3 = 0.$$

Solve by separation of variables:

$$u(x) = \frac{V}{2} \operatorname{sech}^2 \left( \frac{\sqrt{V}}{2}(x + c) \right),$$



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- Nonlinear functions in equation must be (Taylor) expanded as series to identify order  $\epsilon$  terms.
- One can study stability and dispersion of the linearization.
- This approximation becomes invalid when  $w(x, t)$  becomes large enough.

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Now linearize about  $u_0 = 1$  by plugging in  $u(x, t) = 1 + \epsilon w(x, t)$ :

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so that the linearization is now

$$w_t = w_{xx} - w.$$

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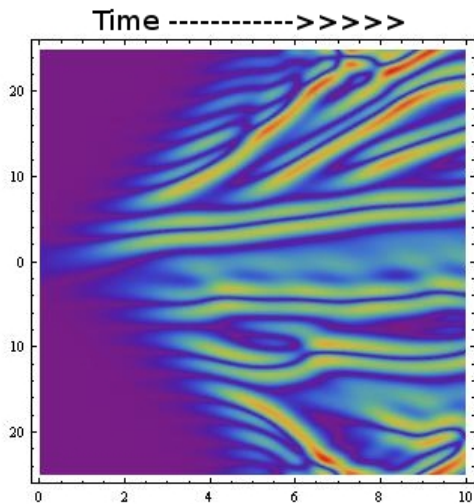
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Dispersion relation of the form  $w = \exp(\sigma t + ikx)$  gives

$$\sigma(k) = -k^4 + k^2.$$

Since  $\sigma > 0$  for  $|k| < 1$ ,  $u = 0$  is unstable.

## Example: Kuramoto-Sivashinsky simulation



### Example 3

A thin liquid film of height  $h(x, t)$  evolves according to the equation

$$h_t = (h^3[-h_{xx} + Ah^{-3}]_x)_x,$$

where  $A$  describes intermolecular forces.

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Linearize about a constant solution  $h(x, t) = h_0$  by setting

$h(x, t) = h_0 + \epsilon w$  and Taylor expand

$$(h_0 + \epsilon w)^3 = h_0^3 + \epsilon 3h_0^2 w + \mathcal{O}(\epsilon^2), \quad (h_0 + \epsilon w)^{-3} = h_0^{-3} - \epsilon 3h_0^{-4} w + \mathcal{O}(\epsilon^2).$$

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Inserting into equation,

$$\epsilon w_t = \left( (h_0^3 + \epsilon 3h_0^2 w)[- \epsilon w_{xx} + h_0^{-3} - \epsilon 3Ah_0^{-4} w]_x \right)_x + \mathcal{O}(\epsilon^2),$$

so that retaining the  $\epsilon$  size terms,

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The corresponding dispersion relation is found from

$w = \exp(\sigma t + ikx)$ , giving

$$\sigma(k) = h_0^3(-k^4 + 3Ah_0^{-4}k^2),$$

Band of unstable wavenumbers  $|k| < h_0^{-2}\sqrt{3A}$  if  $A > 0$ .

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For wave type equation, find dispersion relation

$w(x, t) = \exp(ikx - i\omega t)$ , giving

$$-\omega^2 = -c^2 k^2 - 1, \quad \omega(k) = \pm \sqrt{1 + c^2 k^2}.$$