Multidimensional eigenvalue problems, example # 1

Fourier's coffee cup: model as a disk

$$u_t = D\Delta u, \quad u(a,\theta,t) = u_a, \quad u(r,\theta,0) = u_0,$$

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Now get problem for $w = u - u_p$ which we can solve:

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Most general solution to this is just superposition of separated solutions

$$w = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}[A_{nm}\cos(n\theta) + B_{nm}\sin(n\theta)]J_n(\beta_{nm}r/a)e^{-D\beta_{nm}^2 t/a^2}$$

Fourier's coffee cup, cont.

Notice initial condition does not depend on θ, so simplifies to

$$w = \sum_{m=1}^{\infty} A_{0m} J_0(\beta_{0m} r/a) e^{-D\beta_{0m}^2 t/a^2}.$$

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Recall $J_0(\beta_{0m} r/a)$ are orthogonal (with respect to weighted inner product) for different m, thus

$$A_{0m} = \frac{\int_0^a J_0(\beta_{0m} r/a)(u_0 - u_a) r \, dr}{\int_0^a J_0^2(\beta_{0m} r/a) r \, dr}$$

Fourier's coffee cup, cont.

Still too complicated! Only use term with slowest decay ("ground state approximation")

$$w \approx A_{01} J_0(\beta_{01} r/a) e^{-D\beta_{01}^2 t/a^2}.$$

It follows that temperature in center is

$$u(0,t) = u_a + w(0,t) \approx u_a + (u_0 - u_a) e^{-D\beta_{01}^2 t/a^2}$$

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For $a = 3cm$, $D = .001cm^2/sec$, $\beta_{01} = 2.404$, exponential decay rate is $\exp(-t/t_c)$ where $t_c = D\beta_{01}^2/a^2 \approx 1000sec$.

Example # 2: Fourier's Doughnut

Problem: find fundamental (smallest) frequency for wave equation

$$u_{tt} = c^2 \Delta u$$

on an annulus $1 < r < 2$, subject to boundary conditions $u(1, \theta, t) = 0 = u(2, \theta, t)$.

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Recall separated solutions $u = T(t)v(r, \theta)$ solve $T'' = -c^2 \lambda T$ and $\Delta v = -\lambda v$. Since $T = \cos(c\sqrt{\lambda} t)$ and $\sin(c\sqrt{\lambda} t)$, frequencies are $c\sqrt{\lambda}$. We therefore want the smallest eigenvalue.

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Separation $v = \Theta(\theta)R(r)$ leads to $\Theta = \cos(n\theta)$ and $\sin(n\theta)$ as before. For each n, R solves the Bessel equation

$$r^2 R'' + rR' + (\lambda r^2 - n^2)R = 0.$$

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In this case, we cannot omit the solutions which are singular at the origin, so

$$R(r) = c_1 J_n(\sqrt{\lambda}r) + c_2 Y_n(\sqrt{\lambda}r)$$

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Eigenvalues are selected by imposing boundary conditions:

$$0 = c_1 J_n(\sqrt{\lambda}) + c_2 Y_n(\sqrt{\lambda}), \quad 0 = c_1 J_n(2\sqrt{\lambda}) + c_2 Y_n(2\sqrt{\lambda}).$$

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This linear system has nonzero solutions if determinant is zero:

$$J_n(\sqrt{\lambda}) Y_n(2\sqrt{\lambda}) = J_n(2\sqrt{\lambda}) Y_n(\sqrt{\lambda})$$

which is better written as intersection point of graphs

$$Q_n(\sqrt{\lambda}) = Q_n(2\sqrt{\lambda}), \quad Q_n(x) = \frac{J_n(x)}{Y_n(x)}$$

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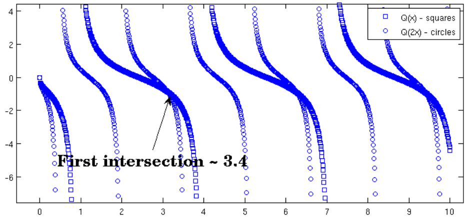

Smallest eigenvalue is therefore $\lambda \approx 3.4^2$, and $\omega = 3.4c$ is the fundamental frequency.

Example # 3: Resonance in forced oscillations

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Look for particular solution which has spatial dependence expanded in eigenfunctions

$$u_p = \cos(\omega_0 t) \sum_{k=1}^{\infty} A_k v_k(x, y)$$

Resonance in forced oscillations, cont.

Plug into equation (using the fact that $\Delta v_k = -\lambda_k v_k$) to get

$$\sum_{k=1}^{\infty} A_k(\lambda_k c^2 - \omega_0^2) v_k(x, y) = 1.$$

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$$\sum_{k=1}^{\infty} A_k(\lambda_k c^2 - \omega_0^2) v_k(x, y) = 1.$$

Just an orthogonal expansion of eigenfunctions, so taking inner products with each eigenfunction gives

$$A_k = \frac{1}{\lambda_k c^2 - \omega_0^2} \frac{\langle v_k, 1 \rangle}{\langle v_k, v_k \rangle}.$$

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Resonance: A system forced with an oscillation near one of its internal frequencies results in a large amplitude response.

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In this case, this means that the particular solution is approximately

$$u_p \approx A_K \cos(\omega_0 t) v_K(x,y).$$

Resonance "picks out" eigenfunction w/ frequency near ω_0.

Resonance in forced oscillations, cont.

Resonance for a disk: Video demonstration

Resonance for a square plate: Video demonstration