Distributional derivatives in higher dimensions

In higher dimensions, distributional derivatives (gradients etc.) are defined using Green's identity: for a smooth function $u(x)$ and $\phi \in \mathcal{D}$, one has

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This motivates the definition of the *distributional Laplacian*:

$$(\Delta u)[\phi] = u[\Delta\phi] = \int_{\mathbb{R}^n} u \, \Delta\phi dx.$$

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$$\Delta f[\phi] = f[\Delta\phi] = \lim_{\epsilon \to 0} \int_{\mathbb{R}^3/B_\epsilon(0)} \frac{\Delta\phi}{|\mathbf{x}|} dx, \quad B_\epsilon(0) = \{|\mathbf{x}| < \epsilon\}.$$

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We can now use Green's identity:

$$\Delta f[\phi] = \lim_{\epsilon \to 0} \int_{\partial B_\epsilon(0)} -\phi \frac{\partial}{\partial n}\left(\frac{1}{|\mathbf{x}|}\right) + \frac{1}{|\mathbf{x}|}\frac{\partial\phi}{\partial n} dx$$

Note that $\hat{n} = -\mathbf{x}/|\mathbf{x}|$, $\partial/\partial n = \partial_r$, and $1/|\mathbf{x}| = 1/\epsilon$ on $\partial B_\epsilon(0)$.

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Since $\partial B_\epsilon(0)$ is the surface of a sphere, we have

$$\int_{\partial B_\epsilon(0)} \phi\, dx \sim 4\pi\epsilon^2\phi(0), \quad \int_{\partial B_\epsilon(0)} \frac{\partial\phi}{\partial n}\, dx = \mathcal{O}(\epsilon^2).$$

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The limit $\epsilon \to 0$ yields $\Delta f[\phi] = -4\pi\phi(0)$; therefore $\Delta f = -4\pi\delta(\mathbf{x})$.

Higher dimensional Green's functions

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$$\Delta u = f(\mathbf{x}), \qquad \lim_{|\mathbf{x}| \to \infty} u(\mathbf{x}) = 0.$$

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Green's function formally solves $\Delta_x G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0)$, which is same as

$$\Delta_x G(\mathbf{x}, \mathbf{x}_0) = 0, \quad \mathbf{x} \neq \mathbf{x}_0, \quad \lim_{|\mathbf{x}|\to\infty} G(\mathbf{x}, \mathbf{x}_0) = 0.$$

with "normalization" condition

$$\int_{\partial B} \nabla_x G(\mathbf{x}, \mathbf{x}_0) \cdot \hat{n} dx = 1.$$

where B is any ball with center x_0.

Green's function in 3D, cont.

Symmetry allows $G = g(r)$, $r = |\mathbf{x} - \mathbf{x}_0|$, so that

$$\frac{1}{r^2}(r^2 g'(r))' = 0 \text{ if } r \neq 0, \quad \lim_{r\to\infty} g(r) = 0.$$

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Integrating twice,

$$g(r) = -\frac{c_1}{r} + c_2, \quad c_2 = 0 \text{ by far-field condition}$$

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Normalization: let B be the unit sphere centered at x_0,

$$1 = \int_{\partial B} \nabla_x G(\mathbf{x}, \mathbf{x}_0) \cdot \hat{n}\, dx = \int_{\partial B} \frac{c_1}{r^2}\, dx = 4\pi c_1,$$

so that $c_1 = 1/4\pi$.

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The Green's function is therefore $G(\mathbf{x}, \mathbf{x}_0) = -1/(4\pi|\mathbf{x} - \mathbf{x}_0|)$ and

$$u(\mathbf{x}) = -\int_{\mathbb{R}^3} \frac{f(\mathbf{x}_0)}{4\pi|\mathbf{x} - \mathbf{x}_0|} dx_0^3.$$

Example: $L = \Delta$ (two dimensions)

$$\Delta u = f, \quad \lim_{r\to\infty}\Big(u(r,\theta) - u_r(r,\theta)\, r \ln r\Big) = 0.$$

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Normalization condition (using $B =$ unit disk)

$$1 = \int_{\partial B} \nabla_x G(\mathbf{x}, 0)\cdot \hat{n}\, dx = \int_{\partial B} \frac{c_1}{r}\, dx = 2\pi c_1,$$

so that $c_1 = 1/2\pi$. Thus the Green's function is $G(\mathbf{x}, \mathbf{x}_0) = \ln|\mathbf{x} - \mathbf{x}_0|/2\pi$, and

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Example: Helmholtz operator

$$\Delta u - u = f(\mathbf{x}), \quad \lim_{r\to\infty} u = 0, \quad u : \mathbb{R}^2 \to \mathbb{R}.$$

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$$1 = \int_{\partial B_r(\mathbf{x}_0)} \frac{\partial_x G}{\partial n}(\mathbf{x},\mathbf{x}_0)\, dx - \int_{B_r(\mathbf{x}_0)} G(\mathbf{x},\mathbf{x}_0) dx \sim \int_{\partial B_r(\mathbf{x}_0)} \frac{\partial_x G}{\partial n}(\mathbf{x},\mathbf{x}_0)\, dx,$$

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as $r \to 0$. It can be shown that $K_0 \sim -\ln(r)$ when r is small, and therefore

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