Distributions and distributional derivatives

Effect of a short pulse

Consider the ordinary differential equation

$$x'(t) + x(t) = f(t), \quad x(0) = 0.$$

Effect of a short pulse

Consider the ordinary differential equation

$$x'(t) + x(t) = f(t), \quad x(0) = 0.$$

Solve using integrating factor $\exp(t)$:

$$x(t) = \int_0^t \exp(s - t) f(s) ds.$$

Effect of a short pulse

Consider the ordinary differential equation

$$x'(t) + x(t) = f(t), \quad x(0) = 0.$$

Solve using integrating factor $\exp(t)$:

$$x(t) = \int_0^t \exp(s - t) f(s) ds.$$

Let $f(t)$ be a pulse of duration $\epsilon \ll 1$ at time t_0. For $t > t_0 + \epsilon$,

$$x(t) = \int_{t_0}^{t_0+\epsilon} \exp(s - t) f(s) ds \approx \exp(t_0 - t) \int_{t_0}^{t_0+\epsilon} f(s) ds = I \exp(t_0 - t),$$

where I is the the integral of $f(t)$.

Effect of a short pulse

Consider the ordinary differential equation

$$x'(t) + x(t) = f(t), \quad x(0) = 0.$$

Solve using integrating factor $\exp(t)$:

$$x(t) = \int_0^t \exp(s - t) f(s) ds.$$

Let $f(t)$ be a pulse of duration $\epsilon \ll 1$ at time t_0. For $t > t_0 + \epsilon$,

$$x(t) = \int_{t_0}^{t_0+\epsilon} \exp(s - t) f(s) ds \approx \exp(t_0 - t) \int_{t_0}^{t_0+\epsilon} f(s) ds = I \exp(t_0 - t),$$

where I is the the integral of $f(t)$.

Can we define a function with unit integral, supported at just a point t_0?

Motivations:

1. Ordinary functions are not enough to describe every physical situation
2. We want to include non-smooth solutions of differential equations

Distributions

Motivations:

1. Ordinary functions are not enough to describe every physical situation
2. We want to include non-smooth solutions of differential equations

Idea: create new class of function-like objects called *distributions* by defining how they “act" on smooth functions.

The delta function

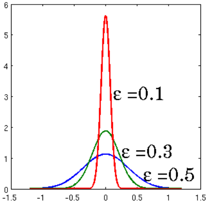
The most basic distribution (or “generalized function") is the so-called δ-function.

The delta function

The most basic distribution (or “generalized function") is the so-called δ-function.

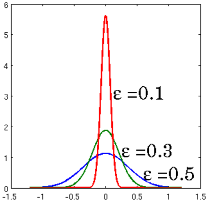
Approximate for $\epsilon > 0$

$$\delta_\epsilon(x) = \frac{1}{\epsilon\sqrt{\pi}} e^{-x^2/\epsilon^2}.$$



The delta function

The most basic distribution (or “generalized function") is the so-called δ-function.



Approximate for $\epsilon > 0$

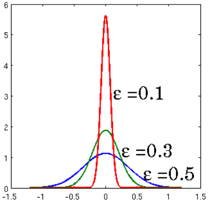
$$\delta_\epsilon(x) = \frac{1}{\epsilon\sqrt{\pi}} e^{-x^2/\epsilon^2}.$$

For any continuous and bounded $f(x)$,

$$\int_{-\infty}^{\infty} f(x)\delta_\epsilon(x - x_0)dx \approx f(x_0) \int_{-\infty}^{\infty} \delta_\epsilon(x - x_0)dx = f(x_0), \quad \epsilon \to 0.$$

The delta function

The most basic distribution (or “generalized function") is the so-called δ-function.



Approximate for $\epsilon > 0$

$$\delta_\epsilon(x) = \frac{1}{\epsilon\sqrt{\pi}} e^{-x^2/\epsilon^2}.$$

For any continuous and bounded $f(x)$,

$$\int_{-\infty}^{\infty} f(x)\delta_\epsilon(x - x_0)dx \approx f(x_0) \int_{-\infty}^{\infty} \delta_\epsilon(x - x_0)dx = f(x_0), \quad \epsilon \to 0.$$

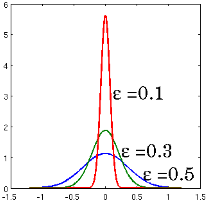
Define the object δ to act as

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)dx = f(x_0).$$

The δ-function “picks out" the value of the function $f(x)$ at x_0.

The delta function

The most basic distribution (or “generalized function") is the so-called δ-function.



Approximate for $\epsilon > 0$

$$\delta_\epsilon(x) = \frac{1}{\epsilon\sqrt{\pi}} e^{-x^2/\epsilon^2}.$$

For any continuous and bounded $f(x)$,

$$\int_{-\infty}^{\infty} f(x)\delta_\epsilon(x - x_0)dx \approx f(x_0)\int_{-\infty}^{\infty} \delta_\epsilon(x - x_0)dx = f(x_0), \quad \epsilon \to 0.$$

Define the object δ to act as

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)dx = f(x_0).$$

The δ-function “picks out" the value of the function $f(x)$ at x_0.

Also define in higher dimensions:

$$\int_{\mathbb{R}^n} f(\mathbf{x})\delta(\mathbf{x} - \mathbf{x}_0)d\mathbf{x} = f(\mathbf{x}_0), \quad \mathbf{x}, \mathbf{x}_0 \in \mathbb{R}^n.$$

Duality and linear functionals

The operation

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)dx = f(x_0).$$

can be regarded as a mapping from a continuous function $f(x)$ to a real number.

Duality and linear functionals

The operation

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)dx = f(x_0).$$

can be regarded as a mapping from a continuous function $f(x)$ to a real number.

Definition: a *linear functional* is a linear mapping from a vector space to the real numbers.

Duality and linear functionals

The operation

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)dx = f(x_0).$$

can be regarded as a mapping from a continuous function $f(x)$ to a real number.

Definition: a *linear functional* is a linear mapping from a vector space to the real numbers.

Examples:
(1) If $g(x) \in C_1(\mathbb{R})$, $g(x) \to g'(0)$ is a linear functional.

Duality and linear functionals

The operation

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)dx = f(x_0).$$

can be regarded as a mapping from a continuous function $f(x)$ to a real number.

Definition: a *linear functional* is a linear mapping from a vector space to the real numbers.

Examples:
(1) If $g(x) \in C_1(\mathbb{R})$, $g(x) \to g'(0)$ is a linear functional.
(2) If $q(x)$ is bounded, integrable,
$f(x) \to \int_{-\infty}^{\infty} q(x)f(x)\,dx$ is a linear functional.

Duality and linear functionals

The operation

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)dx = f(x_0).$$

can be regarded as a mapping from a continuous function $f(x)$ to a real number.

Definition: a *linear functional* is a linear mapping from a vector space to the real numbers.

Examples:
(1) If $g(x) \in C_1(\mathbb{R})$, $g(x) \to g'(0)$ is a linear functional.
(2) If $q(x)$ is bounded, integrable,
$f(x) \to \int_{-\infty}^{\infty} q(x)f(x)\,dx$ is a linear functional.

Should $q(x)$ be regarded as a function, or a *functional*?

Duality and linear functionals

The operation

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)dx = f(x_0).$$

can be regarded as a mapping from a continuous function $f(x)$ to a real number.

Definition: a *linear functional* is a linear mapping from a vector space to the real numbers.

Examples:
(1) If $g(x) \in C_1(\mathbb{R})$, $g(x) \to g'(0)$ is a linear functional.
(2) If $q(x)$ is bounded, integrable,
$f(x) \to \int_{-\infty}^{\infty} q(x)f(x)\, dx$ is a linear functional.

Should $q(x)$ be regarded as a function, or a *functional*?
BOTH! We call this situation *duality*.

Distributions in general

Idea: create new set of objects not as functions, but as continuous linear functionals.

Distributions in general

Idea: create new set of objects not as functions, but as continuous linear functionals.

Precise definition: A *distribution* is a continuous linear functional on the set of infinitely differentiable functions with bounded support (Notated C_0^∞ or simply $\mathcal{D}$). Write

$$d[\phi] : \mathcal{D} \to \mathbb{R}$$

Distributions in general

Idea: create new set of objects not as functions, but as continuous linear functionals.

Precise definition: A *distribution* is a continuous linear functional on the set of infinitely differentiable functions with bounded support (Notated C_0^∞ or simply $\mathcal{D}$). Write

$$d[\phi] : \mathcal{D} \to \mathbb{R}$$

Some facts:

- A continuous function $g(x)$ can be regarded as a distribution by setting $g[\phi] = \int_{-\infty}^{\infty} g(x)\phi(x)dx$.

Distributions in general

Idea: create new set of objects not as functions, but as continuous linear functionals.

Precise definition: A *distribution* is a continuous linear functional on the set of infinitely differentiable functions with bounded support (Notated C_0^∞ or simply $\mathcal{D}$). Write

$$d[\phi] : \mathcal{D} \to \mathbb{R}$$

Some facts:

- A continuous function $g(x)$ can be regarded as a distribution by setting $g[\phi] = \int_{-\infty}^{\infty} g(x)\phi(x)dx$.
- Distributions can be approximated by usual functions: there exists a sequence $d_n(x) \in \mathcal{D}$ so that

$$d[\phi] = \lim_{n\to\infty} \int d_n(x)\phi(x)dx, \quad \text{for all } \phi \in \mathcal{D}.$$

Distributions in general

Idea: create new set of objects not as functions, but as continuous linear functionals.

Precise definition: A *distribution* is a continuous linear functional on the set of infinitely differentiable functions with bounded support (Notated C_0^∞ or simply $\mathcal{D}$). Write

$$d[\phi] : \mathcal{D} \to \mathbb{R}$$

Some facts:

- A continuous function $g(x)$ can be regarded as a distribution by setting $g[\phi] = \int_{-\infty}^{\infty} g(x)\phi(x)dx$.
- Distributions can be approximated by usual functions: there exists a sequence $d_n(x) \in \mathcal{D}$ so that

$$d[\phi] = \lim_{n\to\infty} \int d_n(x)\phi(x)dx, \quad \text{for all } \phi \in \mathcal{D}.$$

- Distributions have integrals:

$$\int_{-\infty}^{\infty} d(x)\phi(x)dx \equiv d[\phi], \quad \text{for any } \phi \in \mathcal{D}.$$

Some examples

Ex. #1: Linear combinations of delta functions are distributions: If $d = 3\delta(x-1) + 2\delta(x)$ then corresponding linear functional is

$$d[\phi] = \int_{-\infty}^{\infty} d(x)\phi(x)dx = 3\phi(1) + 2\phi(0).$$

Some examples

Ex. #1: Linear combinations of delta functions are distributions: If $d = 3\delta(x-1) + 2\delta(x)$ then corresponding linear functional is

$$d[\phi] = \int_{-\infty}^{\infty} d(x)\phi(x)dx = 3\phi(1) + 2\phi(0).$$

Ex. #2: $d[\phi] = \int_0^\infty x\phi'(x)dx$. (check that it's linear!)

Some examples

Ex. #1: Linear combinations of delta functions are distributions: If $d = 3\delta(x-1) + 2\delta(x)$ then corresponding linear functional is

$$d[\phi] = \int_{-\infty}^{\infty} d(x)\phi(x)dx = 3\phi(1) + 2\phi(0).$$

Ex. #2: $d[\phi] = \int_0^\infty x\phi'(x)dx$. (check that it's linear!)
Can it be written in the form $\int_{-\infty}^{\infty} g(x)\phi(x)dx$?

Some examples

Ex. #1: Linear combinations of delta functions are distributions: If $d = 3\delta(x-1) + 2\delta(x)$ then corresponding linear functional is

$$d[\phi] = \int_{-\infty}^{\infty} d(x)\phi(x)dx = 3\phi(1) + 2\phi(0).$$

Ex. #2: $d[\phi] = \int_0^\infty x\phi'(x)dx$. (check that it's linear!)
Can it be written in the form $\int_{-\infty}^{\infty} g(x)\phi(x)dx$? Integrate by parts:

$$d[\phi] = x\phi(x)\Big|_0^\infty - \int_0^\infty \phi(x)dx = \int_{-\infty}^{\infty} \Big(-H(x)\Big)\phi(x)dx,$$

where the "step function" is

$$H(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0. \end{cases}$$

Some examples

Ex. #1: Linear combinations of delta functions are distributions: If $d = 3\delta(x-1) + 2\delta(x)$ then corresponding linear functional is

$$d[\phi] = \int_{-\infty}^{\infty} d(x)\phi(x)dx = 3\phi(1) + 2\phi(0).$$

Ex. #2: $d[\phi] = \int_0^\infty x\phi'(x)dx$. (check that it's linear!)
Can it be written in the form $\int_{-\infty}^{\infty} g(x)\phi(x)dx$? Integrate by parts:

$$d[\phi] = x\phi(x)\Big|_0^\infty - \int_0^\infty \phi(x)dx = \int_{-\infty}^{\infty} \Big(-H(x)\Big)\phi(x)dx,$$

where the “step function" is

$$H(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0. \end{cases}$$

Abuse of notation: $d = -H$.

Distributions as derivatives

Can a non-differentiable function or distribution be differentiated?

Distributions as derivatives

Can a non-differentiable function or distribution be differentiated? We define the *distributional derivative* of $d[\phi]$ to be the distribution $d'[\phi]$ so that

$$d'[\phi] \equiv -d[\phi'].$$

Distributions as derivatives

Can a non-differentiable function or distribution be differentiated? We define the *distributional derivative* of $d[\phi]$ to be the distribution $d'[\phi]$ so that

$$d'[\phi] \equiv -d[\phi'].$$

Idea behind this: if $d[\phi] = \int_{-\infty}^{\infty} g(x)\phi(x)dx$ for smooth $g(x)$, integration by parts gives

$$g'[\phi] = \int_{-\infty}^{\infty} g'(x)\phi(x)dx \equiv -\int_{-\infty}^{\infty} g(x)\phi'(x)dx = -g[\phi'], \quad \text{for all } \phi \in \mathcal{D}.$$

Distributions as derivatives

Can a non-differentiable function or distribution be differentiated? We define the *distributional derivative* of $d[\phi]$ to be the distribution $d'[\phi]$ so that

$$d'[\phi] \equiv -d[\phi'].$$

Idea behind this: if $d[\phi] = \int_{-\infty}^{\infty} g(x)\phi(x)dx$ for smooth $g(x)$, integration by parts gives

$$g'[\phi] = \int_{-\infty}^{\infty} g'(x)\phi(x)dx \equiv -\int_{-\infty}^{\infty} g(x)\phi'(x)dx = -g[\phi'], \quad \text{for all } \phi \in \mathcal{D}.$$

Ex. #1: Compute derivative of step function $H(x)$.

Distributions as derivatives

Can a non-differentiable function or distribution be differentiated? We define the *distributional derivative* of $d[\phi]$ to be the distribution $d'[\phi]$ so that

$$d'[\phi] \equiv -d[\phi'].$$

Idea behind this: if $d[\phi] = \int_{-\infty}^{\infty} g(x)\phi(x)dx$ for smooth $g(x)$, integration by parts gives

$$g'[\phi] = \int_{-\infty}^{\infty} g'(x)\phi(x)dx \equiv -\int_{-\infty}^{\infty} g(x)\phi'(x)dx = -g[\phi'], \quad \text{for all } \phi \in \mathcal{D}.$$

Ex. #1: Compute derivative of step function $H(x)$. The distributional derivative $H'[\phi]$ of H is the rule

$$H'[\phi] = -\int_{-\infty}^{\infty} H(x)\phi'(x)dx = -\int_{0}^{\infty} \phi'(x)dx = \phi(0) = \int_{-\infty}^{\infty} \delta(x)\phi(x)dx.$$

Thus $H' = \delta$.

Distributions as derivatives

Can a non-differentiable function or distribution be differentiated? We define the *distributional derivative* of $d[\phi]$ to be the distribution $d'[\phi]$ so that

$$d'[\phi] \equiv -d[\phi'].$$

Idea behind this: if $d[\phi] = \int_{-\infty}^{\infty} g(x)\phi(x)dx$ for smooth $g(x)$, integration by parts gives

$$g'[\phi] = \int_{-\infty}^{\infty} g'(x)\phi(x)dx \equiv -\int_{-\infty}^{\infty} g(x)\phi'(x)dx = -g[\phi'], \quad \text{for all } \phi \in \mathcal{D}.$$

Ex. #1: Compute derivative of step function $H(x)$. The distributional derivative $H'[\phi]$ of H is the rule

$$H'[\phi] = -\int_{-\infty}^{\infty} H(x)\phi'(x)dx = -\int_{0}^{\infty} \phi'(x)dx = \phi(0) = \int_{-\infty}^{\infty} \delta(x)\phi(x)dx.$$

Thus $H' = \delta$.

Ex. #2: What is the rule implied by the derivative of the δ-function?

Distributions as derivatives

Can a non-differentiable function or distribution be differentiated?
We define the *distributional derivative* of $d[\phi]$ to be the distribution $d'[\phi]$ so that

$$d'[\phi] \equiv -d[\phi'].$$

Idea behind this: if $d[\phi] = \int_{-\infty}^{\infty} g(x)\phi(x)dx$ for smooth $g(x)$, integration by parts gives

$$g'[\phi] = \int_{-\infty}^{\infty} g'(x)\phi(x)dx \equiv -\int_{-\infty}^{\infty} g(x)\phi'(x)dx = -g[\phi'], \quad \text{for all } \phi \in \mathcal{D}.$$

Ex. #1: Compute derivative of step function $H(x)$. The distributional derivative $H'[\phi]$ of H is the rule

$$H'[\phi] = -\int_{-\infty}^{\infty} H(x)\phi'(x)dx = -\int_{0}^{\infty} \phi'(x)dx = \phi(0) = \int_{-\infty}^{\infty} \delta(x)\phi(x)dx.$$

Thus $H' = \delta$.

Ex. #2: What is the rule implied by the derivative of the δ-function?
By definition, $\delta'[\phi] = -\delta[\phi'] = -\phi'(0)$.

Distributions as derivatives: more examples

The n-th derivative of a distribution d is defined to be

$$(-1)^n d[\phi^{(n)}].$$

(this follows from integration by parts n-times).

Distributions as derivatives: more examples

The n-th derivative of a distribution d is defined to be

$$(-1)^n d[\phi^{(n)}].$$

(this follows from integration by parts n-times).

Example: Find rule representing second derivative of $f(x) = |x|$.

Distributions as derivatives: more examples

The n-th derivative of a distribution d is defined to be

$$(-1)^n d[\phi^{(n)}].$$

(this follows from integration by parts n-times).

Example: Find rule representing second derivative of $f(x) = |x|$.
By definition,

$$f''[\phi] = (-1)^2 f[\phi''] = \int_{-\infty}^{\infty} |x|\phi''(x)dx.$$

Distributions as derivatives: more examples

The n-th derivative of a distribution d is defined to be

$$(-1)^n d[\phi^{(n)}].$$

(this follows from integration by parts n-times).

Example: Find rule representing second derivative of $f(x) = |x|$.
By definition,

$$f''[\phi] = (-1)^2 f[\phi''] = \int_{-\infty}^{\infty} |x|\phi''(x)dx.$$

Now integrate by parts and integrate again:

$$\begin{aligned} f''[\phi] &= -\int_{-\infty}^{0} x\phi''(x)dx + \int_{0}^{\infty} x\phi''(x)dx \\ &= -x\phi'(x)|_{-\infty}^{0} + x\phi'(x)|_{0}^{\infty} + \phi(x)|_{-\infty}^{0} - \phi(x)|_{0}^{\infty} = 2\phi(0). \end{aligned}$$

So the second derivative of $|x|$ in the distributional sense is $2\delta(x)$.

Distributions as solutions of equations

Returning to the first differential equation

$$x'(t) + x(t) = f(t), \quad x(0) = 0,$$

with $f(t) = \delta(t - t_0)$ the solution is

$$x(t) = \int_0^t \exp(s - t)\delta(t - t_0) ds = \begin{cases} 0 & t < t_0, \\ \exp(t_0 - t) & t \geq t_0. \end{cases}$$

This solution is discontinuous, so how does is solve the equation?

Distributions as solutions of equations

Returning to the first differential equation

$$x'(t) + x(t) = f(t), \quad x(0) = 0,$$

with $f(t) = \delta(t - t_0)$ the solution is

$$x(t) = \int_0^t \exp(s - t)\delta(t - t_0)ds = \begin{cases} 0 & t < t_0, \\ \exp(t_0 - t) & t \geq t_0. \end{cases}$$

This solution is discontinuous, so how does is solve the equation? Computing a distributional derivative,

$$x'[\phi] = -\int_{-\infty}^{\infty} x(t)\phi'(t)dt = -\int_{t_0}^{\infty} x(t)\phi'(t)dt = x(t_0)\phi(t_0) + \int_{t_0}^{\infty} x'(t)\phi(t)dt.$$

The last integral can be written as

$$-\int_{-\infty}^{\infty} H(t - t_0)\exp(t_0 - t)\phi(t)dt,$$

so

$$x'(t) = \delta(t - t_0) - H(t - t_0)\exp(t_0 - t),$$

and $x'(t) + x(t) = \delta(t - t_0)$ as desired.