RESEARCH STATEMENT

SELIN KALAYCIOĞLU

1. Introduction

My primary areas of research are computational group theory and the representation theory of finite groups. Generally speaking, representation theory is the study of possible ways to realize a group as a matrix group. More specifically, a **representation** ρ of a finite group G over a field F is a homomorphism $\rho\colon G \to GL(V)$ of G into the group $GL(V)$ of invertible F-endomorphisms of a finite dimensional vector space V over F. Any representation ρ of a finite group G over a field F extends to a representation of the group algebra FG, $\rho\colon FG \to \mathrm{End}(V)$ and the vector space V is called an FG-module. The indecomposable direct summands of the regular FG-module are called the **projective indecomposable modules** of G. Computing the projective indecomposable modules for large finite groups is a challenging problem due to the large size of the representations of these groups.

In my dissertation [4], I have developed a new technique for computing the projective indecomposable modules of large finite groups and implemented it in the computer algebra system GAP. Not only is this method more efficient than previous approaches but it may be successfully applied to larger groups. I have used my technique to find the previously unknown projective indecomposable modules of the largest sporadic Mathieu group M_{24} and of the alternating group A_{12} over the field F with characteristic two. My results allow for further analysis of large finite simple groups and their representations. These results also have direct applications in other areas of Algebra; for example, in [2], the authors describe a scheme for computing the Ext-algebra and cohomology ring for a small simple or nearly simple group which is dependent upon knowing the projective indecomposable modules for this group. Similarly computing the cohomology groups of FG-modules via projective resolutions depends on the projective indecomposable modules of G. My method for calculating the projective indecomposable modules presents an opportunity to extend this and other techniques to larger groups. The paper describing my results is in preparation and will be submitted for review soon.

2. The Projective Indecomposable Modules

In this section I give a summary of how one can compute the projective indecomposable modules of a group G and I describe why the known methods were not applicable to groups of very large order. The projective indecomposable modules of G appear as the direct summands of the regular FG-module; however, for large groups G, the regular FG-module is computationally inaccessible since its dimension is the order of G. As an example, consider the sporadic group M_{24} with order 244823040. Storing a matrix for the regular representation of M_{24} over the field with characteristic two would naively require approximately 56 petabytes (approximately 10^{15} bytes), far beyond the capabilities of contemporary machines. There is a categorical equivalence, called **Morita equivalence**, which solves this problem. For two Morita equivalent algebras A and B, there is a correspondence

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between A-modules and B-modules. Algebras equivalent to FG are of particular importance to my research since they provide a means to investigate the regular FG-module.

We first find an algebra which is Morita equivalent to FG. Let $S_1, \ldots, S_r$ be representatives of the isomorphism classes of simple FG-modules, i.e. FG-modules that have only trivial FG-submodules. If e is an idempotent in FG such that $S_i e \neq 0$ for all i, then e is called a **faithful idempotent**. Faithful idempotents are easy to find in the group algebra FG. For a certain subgroup H of G, where the characteristic of F, p, does not divide the order of H, the idempotent $e_H := \frac{1}{|H|} \sum_{h \in H} h$ is faithful and the group algebra FG is Morita equivalent to the algebra $e_H FG e_H$, called the **condensation subalgebra** of FG and H is called the **condensation subgroup** of G.

The $e_H FG e_H$-module Ve_H consists of the fixed points of the action of H on the regular FG-module V, hence the dimension of Ve_H is usually much smaller than the dimension of V. However, any information that we obtain about Ve_H gives rise to information about V via Morita equivalence, for example the projective indecomposable summands of the regular FG-module V is in one to one correspondence with the projective indecomposable direct summands of the regular $e_H FGeH$-module Ve_H. There is one practical problem: If $\{a_1, \ldots, a_m\}$ are generators for the algebra FG there is no general reason why $\{e_H a_1 e_H, \ldots, e_H a_m e_H\}$ should be algebra generators for $e_H FG e_H$. I used a criterion given by Wiegelmann, in his PhD thesis [7], that ensures that a set of elements in $e_H FG e_H$ generates the algebra $e_H FG e_H$.

The generators for the condensation algebra allow us to construct the $e_H FG e_H$-module Ve_H through a process known as **fixed point condensation**. It is important here to note that the matrices corresponding to the generators of the condensed algebra $e_H FG e_H$ have at most $|H|$ many nonzero entries per row. Such matrices are called **row sparse**. In my implementation, I store only the nonzero entries and their positions of the generating matrices, which makes a dramatic reduction in the required memory. Recalling the M_{24} example, we take H to be the subgroup of order 27. Then the algebra $e_H F M_{24} e_H$ has dimension 336224, much smaller than 244823040 and it has 5 generating matrices. However the regular $e_H F M_{24} e_H$-module is still out of reach of what would be reasonable to work with on a computer today. On the other hand storing 5 row sparse matrices corresponding to the algebra generators $e_H g e_H$ with at most 27 nonzero entries per row requires only 215 megabytes.

Having computed the condensed regular $e_H FG e_H$-module Ve_H that decomposes into the projective indecomposable modules of $e_H FG e_H$, we have reduced the problem of finding the projective indecomposable modules of G to finding the direct summands of $e_H FG e_H$. We continue by a theory called **"peakword condensation"** which gives a method for finding the projective indecomposable summands of the regular $e_H FG e_H$-module Ve_H. Let us describe how this theory can be applied to the condensed subalgebra $e_H FG e_H$. Let S be a composition factor of the $e_H FG e_H$-module Ve_H. An idempotant $e \in e_H FG e_H$ is called **S-primitive** if the module $e e_H FG e_H$ has a unique maximal submodule such that the corresponding factor module is isomorphic to S. In general it is a difficult problem to determine such an idempotent, but this may be resolved by considering certain elements of the algebra $e_H FG e_H$ called **peakwords**.

We next define a peakword and then describe the relation between S-primitive idempotents and S-peakwords. For an element $a \in e_H FG e_H$ we denote the kernel and the image of the action of a on the $e_H FG e_H$-module Ve_H by $\mathrm{Ker}_{Ve_H}(a)$ and $\mathrm{Im}_{Ve_H}(a)$, respectively. An element $a \in e_H FG e_H$ is called an **S-peakword** if

$\mathrm{Ker}_T(a) = 0$ for all composition factors T of Ve_H which are not isomorphic to S, and $\dim_F(\mathrm{Ker}_S(a^2))$ is equal to the splitting field of S. Given an S-peakword a, the decomposition $Ve_H = \mathrm{Ker}_{Ve_H}(a^r) \oplus \mathrm{Im}_{Ve_H}(a^r)$ for large $r \in \mathbf{N}$ gives rise to the condensation of Ve_H with a uniquely determined S-primitive idempotent e, such that $Ve_He = \mathrm{Ker}_{Ve_H}(a^r)$. Moreover in [3] it is proven that a uniformly at random chosen vector from Ve_He will generate the projective indecomposable summand with probability $\frac{1}{|\mathrm{End}_{e_HFGe_H}(S)|}$, where $\mathrm{End}_{e_HFGe_H}(S)$ is the splitting field of S.

There are existing implementations that calculate the S-peakword and find a basis of $\mathrm{Ker}_V(a^r)$, namely a basis of Ve for a given FG-module V. However these implementations have limitations because they are not compatible with sparse matrices and they are limited in the size of representations which they can handle.

The peakwords are just sums of products of generating matrices for e_HFGe_H. Since I store the generating matrices in row sparse format, creating the peakwords in the regular representation is out of question because there is no guarantee that the product of two row sparse matrices is row sparse. Hence a different method is required for finding a vector in $\mathrm{Ker}_{Ve_H}(a^r)$ which will generate the projective indecomposable summand of e_HFGe_H where the sparseness of the generating matrices are never lost.

In my dissertation, I create and implemented algorithms that compute a single vector in $\mathrm{Ker}_{Ve_H}(a^r)$ instead of a basis of $\mathrm{Ker}_{Ve_H}(a^r)$. The **order polynomial** of a vector v plays an important role for computing such a vector, i.e. a polynomial $p(x)$ such that for a fixed linear transformation $T \in \mathrm{End}_F(Ve_H)$, $vp(T) = 0$. So if $p(x) = x^r q(x)$ is the order polynomial of a random vector $v \in Ve_H$ for a fixed S-peakword $a \in e_HFGe_H$ then $vp(a) = vq(a)a^r = 0$. This implies that $vq(a)$ is a vector in the $\mathrm{Ker}_V(a^r)$. Using my algorithms I calculate the projective indecomposable modules of G with the following steps:

(1) Use GAP to generate a random vector v in Ve_H.
(2) Compute the generating matrices for e_HFGe_H in sparse format.
(3) Define the S-peakword a corresponding to the simple e_HFGe_H-module S in terms of the sparse generating matrices.
(4) Calculate the order polynomial $p(x)$ of v for a.
(5) Divide $p(x)$ by x^k to obtain the quotient $q(x)$ as described above. .
(6) Compute $vq(a)$.
(7) Compute the e_HFGe_H-module generated by $vq(a)$.

In the final step described above, I can verify that $vq(a)$ really generates the projective indecomposable module corresponding to the simple module S. If $vq(a)$ does not generate a module of dimension equal to the dimension of the condensed e_HFGe_H-projective indecomposable module then I go to first step and choose a new random vector.

3. Future Research Plans

Finding the projective indecomposable modules for large simple groups using my algorithm is an ongoing project. For example, the projective indecomposable modules for the McLaughlin group McL over F_3 and for the Held group He over F_2, F_3 and F_5 are still not known and can be found using my algorithms.

As mentioned earlier the computational techniques described above have applications in homological algebra, such as finding Ext-algebras, cohomology ring for large group algebras and computing the cohomolgy groups of FG-modules. I would like to find the unknown Ext-algebras and cohomology ring for large group algebras over fields of various characteristics. The technique described in [2] for finding the Ext-algebras of group algebras FG has three steps: 1. finding the basic algebra (i.e., the smallest algebra that is Morita equivalent to FG), 2. computing the projective resolutions for simple FG-modules, and 3. computing the cup products. Steps 1 and 2 are dependent upon knowing the projective indecomposable modules for G. By using my algorithm, I will be able to compute basic algebras for larger groups. I am also interested in implementing algorithms to compute the first and second cohomology groups of FG-modules for large finite simple groups G.

In my technique for finding the projective indecomposable modules of a large group G, I was restricted to use projective FG-modules. In [6], Robinson describes a different approach for finding the projective indecomposable modules of G by using induced modules, where these induced modules are not necessarily projective. This approach has not been used for computational purposes; I would like to implement it as a potentially more-efficient algorithm.

Condensation is an important tool in modular representation theory. For example, it is used to find submodule lattices [3] and to determine the socle and radical series of FG-modules [5]. I am interested in a problem considered by D. Benson in [1]. He looked at indecomposable direct summands of permutation modules and defined "characters" for them. These direct summands are known as trivial source modules. Tables of these "characters" have been calculated for small groups. Investigating methods for systematically calculating these tables using basic algebras for larger groups is another project that I will work on.

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