$T(d)\ T(cd)$

(5) a) True

b) False

c) ~~False~~

(6) T_1 : reflection in the y axis $[T_1] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

T_2 : stretching by a factor of 2 in the x direction

$$T_2\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \end{bmatrix} \qquad [T_2] = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

T_3 : clockwise rotation by 45°

$$[T_3] = \begin{bmatrix} \cos 45 & \sin 45 \\ -\sin 45 & \cos 45 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$T = T_3 \circ T_2 \circ T_1$$

$$[T] = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2/\sqrt{2} & 1/\sqrt{2} \\ 2/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Image of $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ under T

$$[T]v = \begin{bmatrix} -2/\sqrt{2} & 1/\sqrt{2} \\ 2/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 3/\sqrt{2} \end{bmatrix}.$$

(7) $\frac{1}{2}\left[T\begin{pmatrix} 1 \\ 1 \end{pmatrix} + T\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right] = T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 \\ 8 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$

$\frac{1}{2}\left(T\begin{pmatrix} 1 \\ 1 \end{pmatrix} - T\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$

Matrix rep of $[T] = \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix}$.

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$\begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} x+y \\ 4x-y \end{bmatrix}.$$

(8) Suppose $\exists$ scalars c_1, c_2 s.t $c_1 v + c_2 T(v) = 0$.

Apply T to both sides.

$$T(c_1 v) + T(c_2 T(v)) = T(0)$$

$$c_1 T(v) + c_2 T^2(v) = 0 \quad \text{(since } T \text{ is a linear transf.)}$$

It's given that $T^2(v) = 0$ therefore $c_1 T(v) = 0$ also since $T(v) \neq 0$ $c_1 = 0$.

If $c_1 = 0$ then $c_2 T(v) = 0$ implies that $c_2 = 0$ since $T(v) \neq 0$