

ALGORITHMIC METHODS IN MODULAR REPRESENTATION THEORY

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1. INTRODUCTION

The main purpose of representation theory is the study of concrete realizations of an abstract algebra. The theory of group representations was developed by G. Frobenius in the end of the last century. Later in 1929 Noether used the notion of the group algebra and explained the relation between the structure theory of algebras and the representation theory of finite groups. This was an important step in the development of representation theory since one could apply the results of representation theory of algebras and module theory to the case of finite groups. Another major development in representation theory was made by R. Brauer when he extended the representation theory of finite groups from fields of characteristic zero to fields of finite characteristic, which is called modular representation theory of finite groups.

The purpose of this paper is to study modular representations, to understand the different ways of looking at a representation, and describe some methods which are used to prove whether a given representation is irreducible, and if not to find its decomposition into irreducible representations.

Date: 6 December 2006.

In Chapter 2, we will introduce some notation, give basic definitions and major theorems, in Chapter 3, the Norton Criterion and the generalized Norton Criterion are described. In Chapter 4, we work out some examples, to show how the generalized Norton Criterion works. In Chapter 5, a method, called condensation, will be introduced.

2. NOTATION AND BASIC DEFINITIONS

In this paper we will be mainly interested in finite-dimensional algebras, their representations and their modules. Therefore in this chapter we will give an introduction to the basic terminology.

Definition 2.1. *Let R be a commutative ring with identity. An R -algebra A is a ring with identity together with a ring homomorphism $f: R \rightarrow A$ mapping 1_R to 1_A such that the subring $f(R)$ is contained in the center of A .*

With this definition in mind one can see that if A is an R -algebra then A has a natural left and right unitary R -module structure defined by $r \cdot a = a \cdot r = f(r)a$, where $f(r)a$ is just the product in the ring and it is the same as $af(r)$ since $f(r)$ is contained in the center of A by assumption. We should remark that this natural structure of an R -algebra A leads us to an alternate definition.

Remark: Suppose that A is an R -algebra. Then A is a ring with identity that is a left unitary R -module satisfying $r \cdot (ab) = (r \cdot a)b = a(r \cdot b)$ for all $r \in R$ and $a, b \in A$. (Note that these are all equal to the product $f(r)ab$ in the ring A). Conversely, these conditions on a ring A define an R -algebra, and are used in some books as the definition of an R -algebra.

In this paper we will mainly deal with F -algebras where F is a field. In this case an algebra A over a field F is a ring A with an identity which is at the same time a vector space over F . Moreover the scalar multiplication in the vector space and the ring multiplication are required to satisfy the axiom

$$\alpha(ab) = (\alpha a)b = a(\alpha b), \quad \alpha \in F, a, b \in A.$$

A subring of A which is also an F -subspace of A is called a subalgebra of A .

An important example for an F -algebra is the group algebra of an arbitrary finite group G over F , which plays an essential role in the modular representation theory.

Definition 2.2. *Let G be a finite group. We consider all formal sums*

$$\sum \alpha_g \cdot g, \quad \alpha_g \in F, g \in G$$

two such expressions being regarded as equal if and only if they have the same coefficients. The operations on the formal sums are defined as follows:

$$\sum \alpha_g g + \sum \beta_g g = \sum (\alpha_g + \beta_g) g$$

and

$$\left(\sum_{g \in G} \alpha_g g \right) \left(\sum_{h \in G} \beta_h h \right) = \sum_{g, h \in G} \alpha_g \beta_h gh = \sum_{t \in G} \gamma_t t,$$

where

$$\gamma_t = \sum_{g \in G} \alpha_g \beta_{g^{-1}t}.$$

Finally we define

$$\alpha \left(\sum \alpha_g g \right) = \sum \alpha \alpha_g g, \quad \text{for all } \alpha \in F.$$

With these operations, it is easy to check that the set of all formal sums forms an algebra FG , called the **group algebra** of G over F .

Now we are ready to define a group representation and an algebra representation. Before we give these definition we want to introduce some notation, which will be used throughout the paper.

If F is a field and V is an n -dimensional vector space over F , we write $GL(V)$ for the general linear group of all invertible linear transformations of V . Let $T \in GL(V)$ and let $B = \{v_1, v_2, \dots, v_n\}$ be a basis of V . For each $j \in \{1, 2, \dots, n\}$, we write the image of v_j under T in terms of the basis B :

$$(v_j)T = \sum_{i=1}^n \alpha_{ji} v_i.$$

Let $M_B(T) = (\alpha_{ji})$ be the $n \times n$ matrix whose j, i entry is α_{ji} , i.e. use the coefficients of the v_i 's in the above computation of $(v_j)T$ for the j^{th} row of this matrix. The matrix $M_B(T)$ is called the **matrix of T with respect to the basis B** . The mapping $T \rightarrow M_B(T)$ maps $GL(V)$ onto the invertible $n \times n$ matrices, which we denote by $GL(n, F)$.

Definition 2.3. Let G be a finite group and V a vector space over F . An **F -representation** of a group G is a homomorphism T from G to $GL(V)$. If a basis B for V is chosen then the composite map

$$G \xrightarrow{T} GL(V) \rightarrow GL(n, F)$$

is called an **F -matrix representation** and it will be denoted by \hat{T} .

One of the main purpose of our paper is to present some algorithms which determine whether a given representation is irreducible. Let us define an irreducible representation.

Definition 2.4. If T is a representation of G on V , then a subspace W of V is called **T -invariant** if $WT(x) \subseteq W$ for all $x \in G$. If V has a proper nonzero T -invariant subspace W then T is called **reducible**, otherwise T is called **irreducible**, or **simple**. If T is an F representation of G and T is equivalent with a direct sum of irreducible F representations then T is called **completely reducible**, or **semisimple**.

Remark: Note that every irreducible representation is completely irreducible.

The centralizer of a representation plays an important role in the theory of group representations. Some facts about the centralizer will be used in the second chapter.

Definition 2.5. If T is a representation of G on V , define the **centralizer** of T to be the algebra of all linear transformations $A: V \rightarrow V$ for which $AT(x) = T(x)A$ for all $x \in G$. We write $C(T)$ for the centralizer of T . The centralizer $C(\hat{T})$ of a corresponding matrix representation \hat{T} is the algebra of those matrices over F that commute with all $\hat{T}(x)$, for all $x \in G$.

One of the first nontrivial results, fundamental to the study of representations over fields of characteristic zero, is Maschke's Theorem.

Theorem 2.6. Suppose that G is a finite group, F is a field and $\text{char}(F)$ does not divide $|G|$. Then every F -representation T of G is completely reducible.

Proof. For a proof see [1, p.99]. □

Definition 2.7. Let A be a finite dimensional algebra over a field F and V a finite dimensional vector space over F . An **F -representation of A** is an algebra homomorphism

$$T: A \rightarrow \text{End}_F(V),$$

that is a mapping which satisfies

$$T(a + b) = T(a) + T(b), \quad T(ab) = T(a)T(b), \quad T(\alpha a) = \alpha T(a),$$

where $a, b \in A$ and $\alpha \in F$.

Now we will show how an F -representation of G yields an F representation of the group algebra FG and vice versa. Let T be a representation of G . Then there is a unique way to extend T to a representation T^* of FG , namely

$$T^*\left(\sum \alpha_g g\right) = \sum \alpha_g T(g).$$

Conversely, every representation of FG , upon restriction to G , yields a representation of G . There is thus a one to one correspondence between F -representations of the group G and representations of the group algebra FG .

Now we are ready to discuss the correspondence between representations of G and FG -modules. Before we start with the discussion let us define an A -module and give an example.

Definition 2.8. *Let A be an F -algebra. A vector space V over F is a **right A -module** if for each $a \in A$ and $v \in V$, a product $v \cdot a \in V$ is defined and satisfies the following rules:*

$$\begin{aligned} v \cdot (a + b) &= va + vb \\ (v + w) \cdot a &= v \cdot a + w \cdot a \\ v \cdot (ab) &= (v \cdot a) \cdot b \\ v \cdot 1_A &= v \end{aligned}$$

for all $a, b \in A$ and for all $v, w \in V$.

Example: The most important example of a right A -module over an algebra A is the vector space A itself, where the module product is the usual ring multiplication in A . This module will be denoted by A_A and is called the **right regular module** of A . Note that the submodules of the right regular module A_A are the right ideals of A .

Definition 2.9. *An A -module V is called **simple**, or **irreducible**, if the only A -submodules of V are 0 and V .*

Let G be a group, and let $T: G \rightarrow GL(V)$ be a representation of G . For each $v \in V$ and $g \in G$, we define the product $vg \in V$ according to the rule

$$vg = vT(g).$$

It is simple to check that the product vg has the above properties, hence any representation of G yields an FG -module. Suppose now that conversely we are given an FG -module V . We obtain a representation of G as follows. Since V is an FG -module, it is an F -module, i.e. it is a vector space over F . For each $g \in G$, we obtain a mapping defined as $T(g): V \rightarrow V$ by setting

$$vT(g) = vg$$

where $g \in G$ and $v \in V$. Then it is straightforward to check that T is a representation of FG .

This discussion shows that there is a bijection between FG -modules and the pairs (V, T) , where T is a representation of G . Also we note

that under the correspondence we described above, an irreducible representation T of G corresponds to a simple FG -module.

3. DIRECT METHODS FOR STUDYING REPRESENTATIONS

The basic algorithmic problems we plan to study in this chapter are the following:

- (1) Decide whether an A -module is simple, i.e. whether an A -representation is irreducible.
- (2) Find a nontrivial A -submodule of a reducible A -module.

These questions are answered by the Meat-Axe. The Meat-Axe is a practical algorithm, first introduced by Richard Parker [3], for testing finite dimensional modules over finite fields for irreducibility, and for finding explicit submodules in the reducible case.

3.1. The Norton Criterion. In this section we will introduce the Norton Criterion, which is the essential tool for proving the irreducibility of a given A -module.

Remark: If the efficiency of such an algorithm does not concern us, then there is an obvious way of proving the irreducibility of an A -module. We can run through all nonzero vectors $v \in V$, determine a basis for vA , where vA is the smallest A -submodule containing v and check whether vA is a proper submodule of V . However this method depends heavily on the field size and the dimension of V . We will see that this is essentially unnecessary.

The algorithm which determines a basis for vA is called the **spinning algorithm**. Since it is an important part of the Norton Criterion we want to describe how the algorithm works.

The Spinning Algorithm:

Input: A finite dimensional A -module V , the corresponding matrix representation δ of A in terms of matrices $\{A_1 = \delta(a_1), \dots, A_k = \delta(a_k)\}$ for a generating set $\{a_1, \dots, a_k\}$ of A , and $v \in V$.

Calculation:

Set $B = \{v\}$

+ For all $b \in B$ do

- For all $A_i \in \{A_1, \dots, A_k\}$ do

* If bA_i is not in the $\text{span}(B)$ then add bA_i to B .

* End if

- Next A_i

+ Next b

Output: A basis B for the A -submodule vA .

From the algorithm, we see that since V is a finite dimensional A -module, B can be obtained in finitely many steps and $\text{Span}(B) = vA$ is the smallest A -submodule containing v .

Let us recall some basic definitions before we state the Norton criterion.

Definition 3.1. *Let V be a finite dimensional F -vector space. Let $\text{Hom}_F(V, F)$ be the **dual space** denoted by V^* . If V is a right A -module for the F -algebra A then V^* is a left A -module in the following way: For $\lambda \in V^*$ and $a \in A$ we define $a \cdot \lambda(v) := \lambda(v \cdot a)$ for all $v \in V$. If W is an F -subspace of V , then we define the **annihilator** W^0 of W in V^* to be $W^0 = \{\lambda \in V^* \mid \lambda(w) = 0\}$ for all $w \in W$. Note that if W is a right A -submodule of V , then W^0 is a left A -submodule of V^* .*

In the Norton Criterion we will use the following facts about dual vector spaces.

Theorem 3.2. *Let V be an A -module and let W be an A -submodule of V . Then $V^*/W^0 \cong W^*$ as left A -modules and $(V^*)^* \cong V$ as right A -modules. If M is an F -vector space and $\phi \in \text{Hom}(V, M)$, then we denote by $\phi^* \in \text{Hom}(M^*, V^*)$ the linear map defined by $\phi^*(\lambda)(v) = \lambda(\phi(v))$ for all $\lambda \in M^*$ and $v \in V$. The kernel of ϕ^* is given by the annihilator of the image of ϕ .*

Proof. See for example [?]. □

Now we are ready to state the Norton Criterion.

Theorem 3.3. *(The Norton Criterion) Let A be an F -algebra, let V be an A -module and let $\delta : A \rightarrow \text{End}_F(V)$ be the corresponding representation of A . Moreover let $a \in A$ with $0 < \text{Ker}(\delta(a)) < V$. Then δ is an irreducible representation of A if and only if*

- (1) *For all nonzero elements $v \in \text{Ker}(\delta(a))$ it holds that $vA = V$,*
- (2) *There exists a nonzero element $v^* \in \text{Ker}(\delta^*(a))$ with $Av^* = V^*$.*

Proof. \Rightarrow : If δ is an irreducible representation then the only A -submodules of V are 0 and V , which implies that V^* is a simple left A -module, hence (1) and (2) follow trivially.

\Leftarrow : Suppose there exists a nontrivial A -submodule W . Then $0 < W^0 < V^*$ in the dual module V^* . We will distinguish two cases:

Case 1: Suppose $W \cap \text{Ker}(\delta(a)) \neq 0$. Then there exists a nonzero vector $v \in W \cap \text{Ker}(\delta(a))$, hence $vA \leq W$.

Case 2: Suppose $W \cap \text{Ker}(\delta(a)) = 0$. This implies that

$$\delta(a)|_W : W \rightarrow W$$

is injective and since W is finite dimensional, it is onto.

Therefore $W\delta(a) = W$. By Theorem 3.2

$$\text{Ker}(\delta(a)^*) = (\text{Im}(\delta(a)))^0.$$

Since $\text{Im}(\delta(a)|_W) \leq \text{Im}(\delta(a))$, $W \leq \text{Im}(\delta(a))$ from which it follows that

$$(\text{Im}(\delta(a)))^0 = \text{Ker}(\delta(a)^*) \leq W^0.$$

Hence for any nonzero vector $v^* \in \text{Ker}(\delta(a)^*)$, it follows that $Av^* \leq W^0$, which completes the proof. \square

The Norton criterion immediately leads to an algorithm that enables us to verify that a given representation of an F -algebra A is irreducible. Let us see how this algorithm works in practice:

Input: We start with a matrix representation δ of A in terms of matrices $\{A_1 = \delta(a_1), \dots, A_k = \delta(a_k)\}$ for a generating set $\{a_1, \dots, a_k\}$ of A ,

- (1) Pick a word a in the generators of A ,
- (2) Determine $\text{Ker}(\delta(a))$,
- (3) If $\text{Ker}(\delta(a))$ is nonzero then for all nonzero vectors $v \in \text{Ker}(\delta(a))$, up to scalars, determine a basis of vA by using the spinning algorithm. If $vA < V$ then return matrices with respect to this basis of vA for the generating system on vA and on V/vA .
- (4) If $vA = V$ for all nonzero $v \in \text{Ker}(\delta(a))$, then determine one vector v^* such that $\delta(a)^*v^* = 0$ and determine a basis of Av^* by using the spinning algorithm with transposed input, namely $\{A_1^T, \dots, A_k^T\}$. If $Av^* < V^*$, then return transposed matrices with respect to this basis for the generating system on Av^* and V^*/Av^* . If $Av^* = V^*$, then return the answer ‘irreducible’.

Output: Either the information that V is irreducible, or a matrix representation of A on an A -submodule W and the quotient space V/W .

Note that in this algorithm only the basic matrix operations such as multiplication, transposing and Gaussian elimination for calculating null spaces are involved. However, Step 3 can be time consuming, since random elements are chosen until we find an element with a nontrivial null space of low dimension, preferably of dimension one. The advantage of finding an element with a null space of dimension one is that it is sufficient to run the third step of the algorithm only for this vector.

However, finding an element with a nonzero null space is not always that easy. Let us state an important theorem, called the **double centralizer theorem** [?, p.405], which will help us in understanding this difficulty.

Theorem 3.4. *Let A be an algebra over a field F and V be a simple A -module. Let*

$$T: A \rightarrow \text{End}_F(V)$$

be the corresponding representation of A . Then $C(C(T(A))) = T(A)$.

Corollary 3.5. *Let V be a simple A -module. If $C(T(A)) = \{\lambda 1 : \lambda \in F\}$, then $C(C(T(A))) = T(A) = \text{End}_F(V)$, hence T is an epimorphism.*

So suppose T is an irreducible representation of A , where A is as in Theorem 3.4, and $C(T(A)) = \{\lambda 1 : \lambda \in F\}$. If the field F has order q , then by Corollary 3.5, the probability of the nullspace of $T(a)$ for $a \in A$ being nontrivial is approximately $1/q$. So when q is large we have to make a large number of choices. However, the next algorithm we will introduce solves this problem.

3.2. The Generalized Norton Criterion. This method is a generalization of the Norton Criterion and it is more efficient since it does not depend so much on the field size [?].

First we want to describe the generalized Norton criterion and then we will compare it with the Norton criterion.

Input: As before we start with a matrix representation δ of A in terms of matrices $\{\delta(a_1), \dots, \delta(a_k)\}$ for a generating set $\{a_1, \dots, a_k\}$ of A ,

- (1) Pick a word a in the generators of A .
- (2) Calculate the characteristic polynomial $c(x)$ of $\delta(a)$.
- (3) Factor $c(x)$ into irreducible factors. For each irreducible factor $p(x)$, do the following:
 - i) Set $b = p(\delta(a))$.
 - ii) Calculate the null space N of b . If $\dim(N(b)) = \deg(p(x))$, then we call $p(x)$ a **good factor** of $c(x)$.
 - iii) Choose a nonzero vector v in $N(b)$ and calculate a basis of the A -submodule vA by using the spinning algorithm. If this is a proper submodule return matrices with respect to this basis of vA for the generating system on vA and on V/vA .
 - iv) If $vA = V$, calculate the null space N of b^T .
 - v) Choose a nonzero vector v^* in $N(b^T)$ and calculate a basis of the left A -submodule Av^* . If this is a proper submodule then return transposed matrices with respect to this basis for the generating system on Av^* and V^*/Av^* .
 - vi) If $p(x)$ is a good factor, then return the answer ‘irreducible’.

Output: Either the information that V is irreducible, or a matrix representation of A on an A -submodule W and the quotient space V/W .

Remark: Note that the only additional calculations compared with the Norton Criterion are calculating and factoring the characteristic polynomial.

In the generalized Norton algorithm we claim that if $p(x)$ is a good factor then it is sufficient to run the algorithm only for one nonzero vector $v \in N(b)$. In other cases, examination of a single vector will not give a conclusive test for irreducibility, but might prove reducibility. If not, another factor of $c(x)$ or another element $\delta(a)$ is selected. Let us explain why this generalized algorithm is much more efficient.

Let $c(x) = p_1(x)^{a_1} p_2(x)^{a_2} \cdots p_n(x)^{a_n}$ be a factorization of $c(x)$ into irreducible polynomials.

Claim 3.6. $\dim(N(p_i(\delta(a)))) = \deg(p_i(x))$ if and only if the multiplicity of p_i in the minimal polynomial of $\delta(a)$ is a_i .

Sketch of proof: Suppose for some i , the multiplicity of $p_i(x)$ in the minimal polynomial of $\delta(a)$ is a_i . Then after a base change $\delta(a)$ is similar to the following matrix

$$\begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & A_n \end{pmatrix}.$$

Here each A_k is similar to

$$\begin{pmatrix} B_{k1} & 0 & \cdots & 0 \\ 0 & B_{k2} & \cdots & 0 \\ \vdots & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & B_{ks} \end{pmatrix},$$

where B_{ki} is the companion matrix for $p_k^{b_{ki}}$ for some $1 \leq b_{ki} \leq a_k$. However the multiplicity of $p_i(x)$ in the minimal polynomial of $\delta(a)$ is a_i therefore A_i consists of one block, i.e. A_i is the companion matrix for $p_i^{a_i}$. Using the fact that $p_i^{a_i}$ is both the characteristic and minimal polynomial of the matrix A_i it follows that

$$\dim N(p_j(A_i)) = \deg(p_i) \cdot \delta_{ij}.$$

Then $p_i(\delta(a))$ is similar to

$$p_i(\delta(a)) = \begin{pmatrix} p_i(A_1) & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & p_i(A_2) & \cdots & \cdots & \cdots & 0 \\ \vdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \cdots & \cdots & p_i(A_i) & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & p_i(A_n) \end{pmatrix}$$

and since $\dim N(p_i(A_i)) = \deg(p_i)$,

$$\dim N(p_i(\delta(a))) = \deg(p_i(x)).$$

The converse direction is trivial.

Hence for any $v \in N(p_i(\delta(a)))$, where p_i is a good factor of the characteristic polynomial,

$$N(p_i(\delta(a))) = \langle v, v\delta(a), \dots, v\delta(a)^{\deg(p_i(x))-1} \rangle$$

is a $\delta(a)$ -invariant $F\langle\delta(a)\rangle$ -module. Since any $v \in N(p_i(\delta(a)))$ can be written as a linear combination of these basis vectors, $vF\langle\delta(a)\rangle = v'F\langle\delta(a)\rangle$ for some $v' \in N(p_i(\delta(a)))$. Therefore under these circumstances it is enough to run the algorithm only for a single vector in $N(p_i(\delta(a)))$.

Remark: A special case occurs if $a_i = 1$, i.e. the multiplicity of $p_i(x)$ is one. Then $p_i(\delta(a))$ will look as follows:

$$p_i(\delta(a)) = \begin{pmatrix} [0]_{k \times k} & \cdots & \cdots & [0]_{k \times k} \\ 0 & A_2(x) & \cdots & 0 \\ \vdots & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & A_n(x) \end{pmatrix},$$

where $k = \deg(p_i(x))$. From the above discussion we can see that $\dim N(p_i(\delta(a))) = k$.

Definition 3.7. $p(x)$ is called a **very good factor**, if its multiplicity in the characteristic polynomial is one.

Remark: Every very good factor is a good factor.

As we promised, we want to compare the Norton Criterion with the generalized Norton Criterion. As usual, let

$$\delta : A \rightarrow \text{End}_F(V)$$

be a representation of an F -algebra A , and let V be an A module.

Recall that in the Norton Criterion, we are interested in $N(\delta(a))$ for $a \in A$, whereas in the generalized Norton Criterion, we look at $N(p(\delta(a)))$, where $p(x)$ is an irreducible factor of the characteristic polynomial of $\delta(a)$. It follows that The Norton Criterion is just a

special case of the generalized Norton Criterion. Suppose the characteristic polynomial of $\delta(a)$ has the irreducible factor $p(x) = x$. Then $N(p(\delta(a))) = N(\delta(a))$ and if $p(x) = x$ is a good factor, then $\dim(N(\delta(a))) = 1$ and it is enough to run the algorithm for one single vector $v \in N(\delta(a))$, otherwise we have to run it for all nonzero vectors in $N(\delta(a))$.

4. EXAMPLES

Let us look at an example and see how the algorithm works. We want to note that we defined a right A -module, where A is an F -algebra and therefore our multiplication is defined as vA . We choose this convention for computer coding purposes.

Example 1: Let $G = S_3$ and $F = F_2$ and let V be an F_2S_3 -module. We are trying to determine all the irreducible F_2S_3 -submodules, i.e. irreducible representations of S_3 over F_2 . We start with a permutation representation δ .

$$\delta : S_3 \rightarrow \text{GL}(V)$$

where

$$(1 \ 2 \ 3) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = A_1,$$

and

$$(1 \ 2) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = A_2,$$

where $A = \{A_1, A_2\}$ is a generating set for the group algebra F_2S_3 .

Step 1: Pick a random element $\delta(a)$, which corresponds to a random element a in the group algebra F_2S_3 using our generating set A . Let

$$\delta(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Step 2: The characteristic polynomial of $\delta(a)$ is $x^3 + 1 = (x+1)(x^2 + x + 1)$, where each factor is irreducible.

Step 3: Let $p(x) = x^2 + x + 1$. Then $b = p(\delta(a)) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

Step 4: The null space of b is $N(b) = \{(k, l, m) \mid k + l + m = 0; k, l, m \in F_2\}$.

Step 5: Let $v_1 = (1 \ 1 \ 0) \in N(b)$. We have to find a basis B of v_1A . We will find B by using the spinning algorithm that we defined in the previous chapter. $v_1A_1 = (1 \ 0 \ 1) = v_2$. Since v_1 and v_2 are

linearly independent, we include v_2 to B . Since $v_2A_2 = v_2$, we can go to the next step, namely we have to apply A_1 and A_2 to v_2 .

$$\begin{aligned}(1 \ 0 \ 1)A_1 &= (0 \ 1 \ 1), \\ (1 \ 0 \ 1)A_2 &= (0 \ 1 \ 1), \\ (0 \ 1 \ 1)A_1 &= (1 \ 1 \ 0), \\ (0 \ 1 \ 1)A_2 &= (1 \ 0 \ 1).\end{aligned}$$

Note that our basis set is $B = \{(1 \ 1 \ 0), (1 \ 0 \ 1)\}$, since $(0 \ 1 \ 1)$ is just the sum of those two vectors, hence linearly dependent.

Let $W = \text{Span}\{(1 \ 1 \ 0), (1 \ 0 \ 1)\}$ which is not the whole space. Hence we have found a submodule of dimension 2, i.e a degree two representation of S_3 .

Let us check whether this representation is irreducible. Since we have a basis for W , we can write explicitly the corresponding degree two representation. Let $v_1 = (1 \ 1 \ 0)$ and $v_2 = (1 \ 0 \ 1)$. Then

$$\begin{aligned}v_1A_1 &= v_2, & v_2A_1 &= v_1 + v_2, \\ v_1A_2 &= v_1, & v_2A_2 &= v_1 + v_2.\end{aligned}$$

Therefore the new representation with respect to this basis will be

$$\begin{aligned}(1 \ 2 \ 3) &\mapsto \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = A_1, \\ (1 \ 2) &\mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = A_2.\end{aligned}$$

We will follow the same steps as above for our new representation to see whether it is irreducible.

Let $\delta(a) = A_2$. Then the characteristic polynomial of $\delta(a)$ is $(x + 1)(x + 1)$. Let $p(x) = x + 1$.

$$b = p(\delta(a)) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Hence the null space of b , $N(b) = \{(k \ 0) | k \in F_2\}$. Let $v = (0 \ 1) \in N(b)$. Then

$$vA_1 = (1 \ 1).$$

So $B_1 = \{(1 \ 0), (0 \ 1)\}$ is a basis for $vA = V$. By the algorithm we can not conclude yet that this degree 2 representation is irreducible. We have to follow the same steps for the dual space V^* .

The null space $N(b^T) = \{(0 \ k) | k \in F_2\}$. Repeating the same steps as above we find that $B_2 = \{(1 \ 0)^T, (0 \ 1)^T\}$ is a basis for $A^*v = V^*$. Therefore we can conclude that this degree 2 representation of S_3 over F_2 is irreducible.

Since we started with a degree 3 representation, i.e. a module of dimension 3, and found a degree 2 irreducible representation we will get a quotient of dimension 1.

Recall that $W = \text{Span}\{(1 \ 1 \ 0), (1 \ 0 \ 1)\}$. Then the complement of W , $\hat{W} = \{(k \ k \ k) | k \in F_2\}$ is the 1 dimensional irreducible submodule, corresponding to the trivial representation.

To summarize, we found 2 irreducible representations of S_3 over F_2 . The trivial representation and a degree 2 irreducible representation. The following theorem tells us actually that these are the only irreducible representations of S_3 over F_2 up to equivalence.

Theorem 4.1. *Let G a finite group and F a field with characteristic p . If for all irreducible representations T of FG , $C(T) = \{\lambda 1 : \lambda \in F\}$, then the number of nonisomorphic simple FG -modules equals the number of conjugacy classes of p -regular elements of G , where an element $g \in G$ is **p -regular** if p does not divide the order of g .*

Proof. For a proof see [?, p.587]. □

Example 2: Now we will look at the irreducible representations of S_3 over F_3 . We will see how changing the underlying field effects the whole problem even in this small example. Let

$$\delta : S_3 \rightarrow GL(V)$$

be a representation of S_3 where

$$(1 \ 2 \ 3) \mapsto A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

and

$$(1 \ 2) \mapsto A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $\delta(a) = A_1$. Then the characteristic polynomial of $\delta(a)$ is $-(x+2)^3$. Let $p(x) = x+2$ and

$$b = p(\delta(a)) = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

The null space of b is, $N(b) = \{(k \ k \ k) | k \in F_3\}$. Let $v \in N(b) = (1 \ 1 \ 1)$. Then vA is a one dimensional irreducible subspace.

Now let $\delta(a) = A_2$. The characteristic polynomial of $\delta(a)$ is $-(x - 1)^2(1 + x)$. If we choose $p(x) = (x + 1)$ then

$$p(\delta(a)) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Hence $N(p(\delta(a))) = \{k \begin{pmatrix} 2k & 0 \\ 0 & 2 \end{pmatrix} : k \in F_3\}$. Following the algorithm, we see that for $v = \begin{pmatrix} 1 & 2 & 0 \end{pmatrix} \in N(p(\delta(a)))$, vA is a one dimensional subspace which is obviously different than the previous one. By Theorem 4.1, S_3 has two conjugacy classes of 3-regular elements. We have found two representations of degree 1 which are not equivalent and these are the only irreducible representations of S_3 up to equivalence when the ground field is F_3 .

Example 3: Finally we will look at the decomposition of the F_3S_4 -module V into its irreducible submodules. Since we showed the detailed computations of the Meat-Axe in the previous examples, we will skip the details, and rather give the result, which will be used in Chapter 5. Let

$$\delta : S_4 \rightarrow GL_4(F_3)$$

be a permutation representation of S_4 where

$$(1 \ 2 \ 3 \ 4) \mapsto A_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and

$$(1 \ 2) \mapsto A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We choose

$$\delta(a) = A_1.$$

The characteristic polynomial of $\delta(a)$ is $(x - 1)(x - 2)(x^2 + 1)$. Let $p(x) = x^2 + 1$. After lengthy computations, we find that

$$N(p(\delta(a))) = \{(k \ l \ 2k \ 2l) | k, l \in F_3\}.$$

For $v = \begin{pmatrix} 1 & 1 & 2 & 2 \end{pmatrix} \in N(p(\delta(a)))$, it can be shown that $vA < V$ is an irreducible subspace of dimension 3 with basis

$$B = \{(1 \ 1 \ 2 \ 2), (1 \ 2 \ 2 \ 1), (2 \ 1 \ 2 \ 1)\}.$$

Let V_2 be the corresponding irreducible F_3S_4 -submodule of V . If V_1 is the trivial subspace of F_3S_4 -module V , spanned by the vector $v = (1 \ 1 \ 1 \ 1)$, then

$$V = V_1 \oplus V_2,$$

since $V_1 \cap V_2 = 0$.

5. CONDENSATION OF MODULAR REPRESENTATIONS

In the second chapter we introduced some direct methods to analyze a given matrix representation of a finite dimensional algebra A and to decompose it into its composition factors. However, these methods are effective for representations of degree up to about 5000. In this section we will introduce a new computational technique, called condensation, which can be applied to representations of much higher degree[2]. The purpose of condensation is to replace an explicit matrix representation of a group by a related but much smaller representation of an algebra for which it is possible to use the Meat-Axe.

Condensation can only be applied efficiently to certain special FG -modules for which the group action can be described by a compact formula, such as permutation modules, tensor product of modules and exterior powers of modules of small degree [4]. In this paper we will only concentrate on an algorithm which condenses an FG -permutation module V , i.e. if we choose a basis $B = \{m_1, \dots, m_n\}$ of V then for all $1 \leq i \leq n$, there exists some j , $1 \leq j \leq n$, such that

$$m_i g = m_j.$$

Before we start with the condensation algorithm, let us give some background information and introduce some notation.

Lemma 5.1. *Let G be a group, F be a finite field, and V be an FG -module. If $e \in FG$ is an idempotent element, i.e. $e^2 = e$, then Ve is an $eFGe$ -module.*

Proof. The proof is straightforward. □

Definition 5.2. *The $eFGe$ -module Ve obtained in Lemma 5.1 is called the **condensed module** from the FG -module V .*

Now we will list some basic lemmas, which are important for the study of the condensation algorithm.

Lemma 5.3. *If $W \leq V$ as FG -modules, then $We \leq Ve$ as $eFGe$ -modules.*

Proof. Note that $We \subseteq Ve$. The proof follows from Lemma 5.1, since We is an $eFGe$ -module. □

Lemma 5.4. *If $W \leq V$ as FG -modules, then $(V/W)e \cong Ve/We$ as $eFGe$ -modules.*

Proof. Let $f: Ve \rightarrow (V/W)e$. It is straightforward to show that f is an $eFGe$ -epimorphism with $\text{Ker } f = We$. \square

Lemma 5.5. *If $X \leq Ve$ as $eFGe$ -modules, then $X = We$ for some FG -submodule W of V .*

Proof. Let $W = XFG$ be an FG -submodule of V . Since e acts as the identity on Ve we obtain:

$$X = Xe \leq XFGe = We = XeFGe \leq X.$$

Thus $X = We$. \square

Lemma 5.6. *If V is an irreducible FG -module, then Ve is either an irreducible $eFGe$ -module or zero.*

Proof. Suppose Ve is reducible. Then there exists a proper $eFGe$ -submodule X of Ve . By Lemma 5.5, $X = We$ for some FG -module W of V , which contradicts the fact that V is irreducible. \square

The following proposition plays an essential role in the condensation algorithm.

Proposition 5.7. *Let V be an FG -module, and $e \in FG$ be an idempotent element. Then the following holds:*

- (1) *Let $\{X_1, \dots, X_r\}$ be the composition factors of V , then the composition factors of Ve are the nonzero members of the set $\{X_1e, \dots, X_re\}$.*
- (2) *Let $0 = V_0 \subset V_1 \subset \dots \subset V_n = V$ be a composition series of V . Then there is a sequence $0 \leq i_0 \leq \dots \leq i_m \leq n$ such that $0 = V_{i_0}e \subset \dots \subset V_{i_m}e = Ve$ is a composition series of Ve . Moreover*

$$V_{i_k}e/V_{i_{k-1}}e \cong (V_{i_k}/V_{i_{k-1}})e$$

as $eFGe$ -modules.

- (3) *Conversely, let $0 = W_0 \subset W_1 \subset \dots \subset W_m = Ve$ be a composition series of Ve . Then V has a composition series $0 = V_{01} \subset \dots \subset V_{0r_0} \subset V_{11} \subset \dots \subset V_{1r_1} \subset \dots \subset V_{m1} \subset \dots \subset V_{mr_m} = V$ such that $V_{ij}e \cong W_i$.*

Proof. The proof follows directly from the above lemmas. \square

Now we are ready to introduce the condensation algorithm.

The Condensation Algorithm:

Input: A finite group G , a finite field F of characteristic p , and an FG -permutation module V which is to be studied.

- (1) Choose a subgroup H of G , where p does not divide $|H|$. The subgroup H gives rise to an idempotent

$$e = \frac{1}{|H|} \sum_{h \in H} h$$

of the algebra FG , thus to a subalgebra

$$eFGe = \left\{ e \left(\sum_{g \in G} \alpha_g \cdot g \right) e \mid \alpha_g \in F \right\}$$

of FG .

- (2) From the FG -module V , obtain the $eFGe$ -module

$$Ve = \{ m \in V \mid mh = m \text{ for all } h \in H \}.$$

- (3) Use the Meat-Axe described in the previous chapter to analyze the $eFGe$ -module Ve .
- (4) Use the correspondence between FG -modules and $eFGe$ -modules described in Proposition 5.7 to get information about composition factors of V .

Output: The composition factors of the FG -module V .

Remark: The $eFGe$ -module Ve consists of the fixed points of the action of H on V , hence the dimension of Ve is much smaller than the dimension of V . Therefore it is much easier to apply the Meat-Axe to Ve . However, any information that we obtain about Ve gives rise to information about V via Proposition 5.7.

We still have to answer the following questions about the algorithm.

- How do we find a basis of the $eFGe$ -module Ve ?
- How do we find a set $\{\alpha_1, \dots, \alpha_t\}$ so that

$$\{e\alpha_1e, \dots, e\alpha_t e\}$$

generates the algebra $eFGe$, and how do we find the actions of the generators of $eFGe$ on the basis of Ve , which will be used as the input for the Meat-Axe in Step3?

Remark: Although it is easy to write down random sets of group elements which probably generate the $eFGe$, it is not easy to verify that a given set of elements generates $eFGe$. Wiegmann gives an important criterion in his PhD thesis [5], which ensures that a set of

elements in $eFGe$ generates the algebra $eFGe$. This result is beyond the scope of our paper, so we will take this fact for granted.

Let us first introduce some notation which will be used throughout the paper and then we will answer the above questions.

Let $B = \{m_1, \dots, m_n\}$ be a basis of V which exhibits the permutation action of G on V . We index the members of this basis by $S = \{1, \dots, n\}$. From the action of G we obtain a permutation representation of G on S , i.e. for all $1 \leq i \leq n$ there exists some j such that

$$m_i g = m_{ig} = m_j.$$

Let $H \leq G$ where the characteristic of F does not divide $|H|$, and

$$e = \frac{1}{|H|} \sum_{h \in H} h$$

be an idempotent element in the algebra FH , hence in FG .

We can write $S = O_1 \dot{\cup} \dots \dot{\cup} O_r$, where O_i 's are the H -orbits of the action of H on S . Here $\dot{\cup}$ stands for disjoint union.

Proposition 5.8. *Let V be an FG -module with basis $B = \{m_1, \dots, m_n\}$, then*

$$\bar{B} = \{\bar{O}_1, \dots, \bar{O}_r\}$$

is a basis of Ve , where

$$\bar{O}_i = \sum_{o_i \in O_i} m_{o_i}.$$

Proof. It is obvious that \bar{O}_i 's are fixed by all elements of $h \in H$. It is not difficult to show that $\{\bar{O}_1, \dots, \bar{O}_r\}$ is linearly independent and spans Ve . \square

Once we have a basis of Ve , then we can determine the action of ege on the basis of Ve for $g \in G$.

Proposition 5.9. *Let V be an FG -module with basis $B = \{m_1, \dots, m_n\}$ and Ve be the $eFGe$ -module with basis $\bar{B} = \{\bar{O}_1, \dots, \bar{O}_r\}$. For $g \in G$, the action of ege on $\bar{B} = \{\bar{O}_1, \dots, \bar{O}_r\}$ is given by*

$$\bar{O}_i e g e = \sum_{j=1}^r c_{ij} \frac{1}{|O_j|} \bar{O}_j,$$

where

$$c_{ij} = |\{o_i \in O_i \mid o_i g \in O_j\}|.$$

Proof. Let $o_i \in O_i$. Then the product

$$(1) \quad m_{o_i}e = \frac{1}{|H|} \sum_{h \in H} m_{o_i}h = \frac{1}{|H|} |H_{o_i}| \sum_{o_i \in O_i} m_{o_i},$$

where $H_{o_i} = \{h \in H | m_{o_i}h = m_{o_i}\}$. Therefore

$$(2) \quad m_{o_i}e = \frac{1}{|O_i|} \sum_{o_i \in O_i} m_{o_i} = \frac{\bar{O}_i}{|O_i|}$$

Hence using (2),

$$\begin{aligned} \bar{O}_i e g e &= \bar{O}_i g e = \left(\sum_{o_i \in O_i} m_{o_i} \right) g e \\ &= \left(\sum_{o_i \in O_i} m_{o_i} g \right) e \\ &= \left(\sum_{o_i \in O_i} m_{o_i} g \right) e \\ &= \sum_{o_i \in O_i, o_i g \in O_j} \frac{\bar{O}_j}{|O_j|} \\ &= \sum_{j=1}^r c_{ij} \frac{\bar{O}_j}{|O_j|}, \end{aligned}$$

where

$$c_{ij} = |\{o_i \in O_i | o_i g \in O_j\}|.$$

□

Our last task is to find $eFGe$ -submodules of Ve which correspond to FG -submodules of V . Note that Ve always has the $eFGe$ -submodule spanned by the single vector $\bar{O}_1 + \cdots + \bar{O}_r$, which we call the trivial subspace. So Ve is never irreducible when $r > 1$.

Once we have found the basis \bar{B} for the $eFGe$ -module Ve and the actions of $eg_i e$'s on the set \bar{B} for the generators of the algebra $eFGe$, then by using the Meat-axe, we can find $\tilde{v} \in Ve$ such that $\tilde{v}eFGe < Ve$. An important step of the condensation algorithm is to find $v \in V$ corresponding to \tilde{v} such that $vFG < V$. However, we can avoid finding a random element in V since by Lemma 5.5, $\tilde{v}eFGe < Ve$ implies that $\tilde{v}FG < V$, which is the desired proper FG -submodule of V . Writing \tilde{v}

in terms of the basis B of V , we obtain the corresponding vector in V . This stage is sometimes called ‘**uncondensation**’.

Let us summarize:

Suppose we have a submodule of Ve generated by one element $\tilde{v} \in Ve$, and we wish to produce a corresponding submodule of V . Let $We = \tilde{v}eFGe < Ve$ be the submodule of interest. If $\tilde{v} = \sum_i v_i \bar{O}_i$, then by Proposition 5.8, the uncondensation of \tilde{v} is given by

$$(3) \quad v = \sum_i \sum_{j \in O_i} v_i m_j,$$

Hence $W = vFG < V$ is the corresponding FG -submodule.

Now we are ready to apply the condensation algorithm to an example.

Example:

In this example we take

$$G = S_4 = \langle (12), (1234) \rangle,$$

and

$$H = \langle (12), (34) \rangle.$$

Let V be an $F_3 S_4$ -permutation module with basis $B = \{m_1, \dots, m_4\}$. We define

$$e = \frac{1}{4} \sum_{h \in H} h,$$

which is an idempotent element in $F_3 S_4$. Our goal is to condense the $F_3 S_4$ -permutation module, to get an $eF_3 S_4 e$ -module Ve . Then we will apply the Meat-Axe to the latter to get a decomposition of Ve . At the end, we will analyze the relationship between the decomposition of the $F_3 S_4$ -module V and $eF_3 S_4 e$ -module Ve .

Since G permutes B , we label the members of B by $S = \{1, 2, 3, 4\}$. The H -orbits of this action are $O_1 = \{1, 2\}$ and $O_2 = \{3, 4\}$, hence

$$S = O_1 \dot{\cup} O_2$$

By Proposition 5.8,

$$\bar{B} = \{\bar{O}_1, \bar{O}_2\}$$

is a basis of Ve , where

$$\bar{O}_1 = m_1 + m_2, \quad \bar{O}_2 = m_3 + m_4.$$

Using Proposition 5.9, we can calculate the actions of ege on the basis \bar{B} . Since we are dealing with a group of order 24, we worked out the action of ege on the basis \bar{B} for all $g \in S_4$ and we determined generators of the algebra $eF_3 S_4 e$, namely $\{eg_1 e, eg_2 e, eg_3 e\}$, where $g_1 = (12)$, $g_2 = (1234)$ and $g_3 = (13)(24)$.

Let us find the matrices corresponding to the actions of the generators on \bar{B} .

$$\bar{O}_1 e g_1 e = c_{11} \frac{\bar{O}_1}{|O_1|} + c_{12} \frac{\bar{O}_2}{|O_2|} = 1\bar{O}_1 + 0\bar{O}_2,$$

$$\bar{O}_2 e g_1 e = c_{21} \frac{\bar{O}_1}{|O_1|} + c_{22} \frac{\bar{O}_2}{|O_2|} = 0\bar{O}_1 + 1\bar{O}_2.$$

$$\bar{O}_1 e g_2 e = c_{11} \frac{\bar{O}_1}{|O_1|} + c_{12} \frac{\bar{O}_2}{|O_2|} = 2\bar{O}_1 + 2\bar{O}_2,$$

$$\bar{O}_2 e g_2 e = c_{21} \frac{\bar{O}_1}{|O_1|} + c_{22} \frac{\bar{O}_2}{|O_2|} = 2\bar{O}_1 + 2\bar{O}_2.$$

Similarly,

$$\bar{O}_1 e g_3 e = c_{11} \frac{\bar{O}_1}{|O_1|} + c_{12} \frac{\bar{O}_2}{|O_2|} = 0\bar{O}_1 + 1\bar{O}_2,$$

$$\bar{O}_2 e g_3 e = c_{21} \frac{\bar{O}_1}{|O_1|} + c_{22} \frac{\bar{O}_2}{|O_2|} = 1\bar{O}_1 + 0\bar{O}_2.$$

Therefore the actions of eg_1e , eg_2e , and eg_3e are given by the matrices

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

respectively.

As we mentioned, the output for the condensation program will be used as an input for the Meat-Axe. Above we found a representation δ of eF_3S_4e . Recall that in the Meat-Axe, to examine the eF_3S_4e -module, first we have to find the characteristic polynomial of $\delta(a)$, where $a \in eF_3S_4e$. Let $a = eg_2e$.

Hence

$$\delta(a) = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.$$

The characteristic polynomial of $\delta(a)$ is $x(x+2)$. If we choose $p(x) = x$, then

$$p(\delta(a)) = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

So $N(p(\delta(a))) = \{(k \ 2k) | k \in F_3\}$. Let $\tilde{v} = (1 \ 2) \in N(p(\delta(a)))$. By the Meat-Axe, $\tilde{v}eFG_e < Ve$ is a 1 dimensional subspace.

If we denote by \tilde{V}_1 the trivial subspace and $\tilde{v}eFG_e = \tilde{V}_2$, then

$$Ve = \tilde{V}_1 \oplus \tilde{V}_2.$$

Note that $\tilde{v} = 1\bar{O}_1 + 2\bar{O}_2$. Let us find $v \in V$ corresponding to $\tilde{v} \in Ve$. By (3),

$$v = 1(m_1 + m_2) + 2(m_3 + m_4),$$

i.e. $v = (1 \ 1 \ 2 \ 2)$, and $V_2 = vFG < V$ is the irreducible F_3S_4 submodule of dimension 3 that we have found in Chapter 3.

Hence if V_1 is the trivial subspace, then

$$V = V_1 \oplus V_2.$$

We conclude that in our case the composition factors of the F_3S_4 -permutation module V is in one to one correspondence with the composition factors of the eF_3S_4e -permutation module Ve .

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