Completions of $Z_p[\![x]\!]$

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Let $A = \mathbb{Z}_p[\![x]\!]$ in the (x)-adic topology, let $f_n : \mathbb{Z}_p[\![x]\!] \to \mathbb{Z}_p[\![x]\!]/(p,x)^n$ be the quotient map, let $B = \varprojlim_{n \geq 0} \mathbb{Z}_p[\![x]\!]/(p,x)^n$, and let $f : A \to B$ be the induced ring homomorphism. Since the kernel of f_n is $(p,x)^n \supseteq (x)^n$ and $\mathbb{Z}_p[\![x]\!]/(p,x)^n$ is finite, f_n is continuous for all n. Therefore, f is continuous.

Furthermore, that the kernel of f is contained in $\bigcap_{n \geq 0} \ker f_n$ is obvious from looking at the relevant inverse limit diagram. Let $a \in (p,x)^n$. Then $v_x(a) = m$ tells us that the lead coefficient of a is in $(p)^{n-m}$. If this holds for arbitrarily large values of n, then the lead coefficient of a must be 0, hence $a = 0$. So f is injective.

Now, the sequence $\{p^n\}_{n \geq 0}$ converges in B and does not converge in A. Therefore, f cannot be a homeomorphism.

Finally, let $\xi \in B$ and let $\xi_{n,0} \in A$ such that $f_n(\xi_{n,0}) \equiv \xi \mod (p,x)^n B$. Note that modulo $(x,p)^n B$, the constant term of $\xi_{n,0}$ agrees with that of ξ. Hence, they agree modulo p^n, since there are no powers of x involved. Therefore, the constant terms of $\xi_{n,0}$ are a convergent sequence of p-adic integers. Let a_0 be their limit and let $\xi_{n,1}$ be $\xi_{n,0}$ with the constant term replaced with a_0. Then $f_n(\xi_{n,1}) \equiv \xi \mod (p,x)^n B$ for all n, and all $\xi_{n,1}$ have the same constant term. Suppose that we have $\xi_{n,k} \in A$ such that $\xi_{n,k}$ have the same coefficients $(a_0, a_1, \ldots, a_{k-1})$ in degrees less than k and $f_n(\xi_{n,k}) \equiv \xi \mod (p,x)^n B$ for all $n \geq 0$. Now, the kth coefficient of $\xi_{n,k}$ is congruent to that of ξ modulo $(p,x)^n B$, thus it is determined as an element of $\mathbb{Z}_p$ up to $(p)^{n-k}$. Therefore, the kth coefficients of $\xi_{n,k}$ form a convergent sequence of p-adic integers with limit a_k. Letting $\xi_{n,k+1}$ be $\xi_{n,k}$ where the k coefficient is replaced with a_k, we get a sequence in A which has the same coefficients in degrees less that $k+1$ and still satisfies out desired congruence conditions.

Now, let $\zeta = \sum_{k \geq 0} a_n x^k \in A$. By construction, $f_n(\zeta) \equiv \xi \mod (p,x)^n B$ for all n. Therefore, $f(\zeta) = \xi$, and I have shown that f is surjective. In summary, $f : A \to B$ is a continuous isomorphism of rings that is not a homeomorphism. I suspect that such a proof could be extended to the case that $\mathbb{Z}_p$ is replaced by any topologically complete DVR.