

The Affine Algebraic Connection

J. David Taylor

October 3, 2018

(All rings are commutative and unital)

Pick an algebraically closed field

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Parallels

Throughout, let k be an algebraically closed field.
We could let k be ...

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Throughout, let k be an algebraically closed field.
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- ... \mathbb{C} to get the classical geometric picture.

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- ... \mathbb{C} to get the classical geometric picture.
- ... the algebraic closure of $\mathbb{C}(T)$, then we'll study relations between algebraic functions.

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- ... $\overline{\mathbb{Q}_p}$ for some prime number p , to get local, p -adic data.

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- ... $\overline{\mathbb{F}_p}$ for combinatorial, residual arithmetic information.

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Keep in mind whichever example(s) you like best.

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Definition

Let $F_1, \dots, F_r \in k[T_1, \dots, T_n]$

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Definition

Let $F_1, \dots, F_r \in k[T_1, \dots, T_n]$ and let $X \subseteq k^n$ be the solution set of the system of equations

$$F_1(T_1, \dots, T_n) = 0$$

$$\vdots$$

$$F_r(T_1, \dots, T_n) = 0$$

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$$F_1(T_1, \dots, T_n) = 0$$

$$\vdots$$

$$F_r(T_1, \dots, T_n) = 0$$

X is called an *affine algebraic set*.

Examples of an Affine Algebraic Set

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Here are some examples:

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Here are some examples:

- The 0 polynomial has all of k^n as its solution set.

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Here are some examples:

- The 0 polynomial has all of k^n as its solution set.
- $XY - 1$ has a hyperbola in k^2 as its solution set.
- In k^3 , the system

$$x^3 + y^3 + xy^2 + yx^2 - x - y = 0$$

$$x^3 - y^3 + xy^2 - yx^2 - x + y = 0$$

is the union of a cylinder and the line through its center.

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Definition

Let $\mathbb{A}^n(k) := k^n$ with the topology where affine algebraic sets are closed. This is called the *Zariski topology* and $\mathbb{A}^n(k)$ is called *affine n -space*.

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- This is a topology.

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- Every closed set is an affine algebraic set (Hilbert's Basis Theorem.)

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- This is a topology.
- Every closed set is an affine algebraic set (Hilbert's Basis Theorem.)
- If $F \in k[T_1, \dots, T_n]$, then $D(F)$ is defined as the set of points $\mathbf{a} \in \mathbb{A}^n(k)$ such that $F(\mathbf{a}) \neq 0$. This is called a distinguished open set.

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- The distinguished open sets form a basis for the Zariski topology on $\mathbb{A}^n(k)$.

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- $\mathbb{A}^1(k) = k$ with the cofinite topology (finite sets are closed).

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- $\mathbb{A}^1(k) = k$ with the cofinite topology (finite sets are closed).
- In $\mathbb{A}^2(k)$ non-empty closed sets are finite or of the solution of some equation $F(T_1, T_2) = 0$ with $F \in k[T_1, T_2]$, or unions of those.

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- In $\mathbb{A}^n(k)$, non-empty closed sets are finite unions of finite intersections of level sets (of polynomials).

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- Every non-empty open subset of $\mathbb{A}^n(k)$ is dense!

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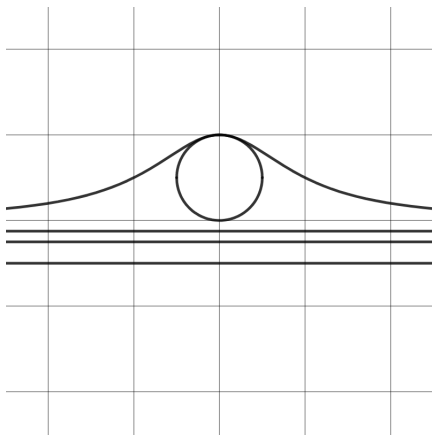
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- Every non-empty open subset of $\mathbb{A}^n(k)$ is dense!
- $GL_n(k)$ is a distinguished open subset of $\mathbb{A}^{n^2}(k)$.

Examples: Zariski Topology

For example, the union of a circle, the Witch of Agnesi, and three lines is an example of a closed set in $\mathbb{A}^2(k)$.



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Affine sets as Topological Spaces

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If X is an affine algebraic set, then $X \subseteq \mathbb{A}^n(k)$. We give it the subspace topology. This topology is still called the Zariski topology.

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Definition

Let X be an affine algebraic set and let $\mathbf{a} \in X$. A function $f : X \rightarrow k$ is *regular at \mathbf{a}* , if \mathbf{a} has an open neighborhood U and there are polynomials $P, Q \in k[T_1, \dots, T_n]$ such that

- 1 Q has no zeros on U , and
- 2 $f(\mathbf{b}) = \frac{P(\mathbf{b})}{Q(\mathbf{b})}$ for all $\mathbf{b} \in U$.

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(i.e. f is locally representable by rational functions)

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(i.e. f is locally representable by rational functions)

Why? Rational functions are the most general functions you can express with $+$, $-$, \times , \div , and we're doing algebra!

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Definition

We say a function $f : X \rightarrow k$ is *regular* if it is regular at every point.

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Definition

We say a function $f : X \rightarrow k$ is *regular* if it is regular at every point.

Note the similarity with the definition of continuous, differentiable, or holomorphic functions on their respective manifolds.

Examples of Regular Functions

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- Any polynomial function is regular.

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- Any polynomial function is regular.
- On $x^2 + y^2 = 1$ in $\mathbb{A}^2(k)$, the function $\frac{x}{1-y}$ defines a regular function on the open set $D(y - 1)$.

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- By the Weak Nullstellensatz, every regular function on a closed set is given by a polynomial.
- So the function $\frac{1}{xy-1}$ on $xy = 2$ in $\mathbb{A}^2(k)$ can be represented by a polynomial.

Regular Morphisms

Definition

Let $X \subseteq \mathbb{A}^n(k)$, $Y \subseteq \mathbb{A}^m(k)$ be affine algebraic sets.

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Definition

Let $X \subseteq \mathbb{A}^n(k)$, $Y \subseteq \mathbb{A}^m(k)$ be affine algebraic sets. A *regular morphism* of algebraic sets $\psi : X \rightarrow Y$ is a function which is given in coordinates by regular functions.

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Let $X \subseteq \mathbb{A}^n(k)$, $Y \subseteq \mathbb{A}^m(k)$ be affine algebraic sets. A *regular morphism* of algebraic sets $\psi : X \rightarrow Y$ is a function which is given in coordinates by regular functions.

More precisely, for every $\mathbf{a} \in X$, there are open neighborhoods

$$\psi(\mathbf{a}) \in V \subseteq Y \text{ and } \mathbf{a} \in U \subseteq \psi^{-1}(V) \subseteq X$$

and rational functions $R_1, \dots, R_m \in k(T_1, \dots, T_n)$ such that

- For all i , the denominator of R_i does not vanish on U , and
- $\psi : U \rightarrow V$ is given by

$$(t_1, \dots, t_n) \mapsto (R_1(t_1, \dots, t_n), \dots, R_m(t_1, \dots, t_n)).$$

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- Regular morphisms of affine algebraic sets are continuous.

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- Regular morphisms of affine algebraic sets are continuous.
- Every regular function on X is a regular morphism $X \rightarrow \mathbb{A}^1(k)$.

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Definition

A *regular isomorphism* of affine algebraic sets is a regular morphism $\psi : X \rightarrow Y$ such that there is a regular morphism $\phi : Y \rightarrow X$ satisfying

- $\phi \circ \psi = \text{id}_X$, and
- $\psi \circ \phi = \text{id}_Y$.

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- The map $\mathbb{A}^1(\overline{\mathbb{F}}_p) \rightarrow \mathbb{A}^1(\overline{\mathbb{F}}_p) : a \mapsto a^p$ is a regular morphism that is a homeomorphism but not an isomorphism.

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- The map $\mathbb{A}^1(\overline{\mathbb{F}_p}) \rightarrow \mathbb{A}^1(\overline{\mathbb{F}_p}) : a \mapsto a^p$ is a regular morphism that is a homeomorphism but not an isomorphism.
- The stereographic projection $\mathbb{A}^1 \rightarrow V(x^2 + y^2 - 1)$ is a regular morphism.

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- The projection $V(y - x^2) \rightarrow \mathbb{A}^1(k) : (a, b) \mapsto a$ is an isomorphism.

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- The projection $V(y - x^2) \rightarrow \mathbb{A}^1(k) : (a, b) \mapsto a$ is an isomorphism.
- The map $\mathbb{A}^1(k) \rightarrow \mathbb{A}^1(k) : a \mapsto c_1 a + c_2$ for $c_1, c_2 \in k$ and $c_1 \neq 0$ is an automorphism. (In fact, these are all the automorphisms of $\mathbb{A}^1(k)$).

Some algebra

- A *k*-algebra *A* is a ring homomorphism $k \rightarrow A$.

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- A *k*-algebra *A* is a ring homomorphism $k \rightarrow A$. That is, *A* is a ring that has *k* as a subring and *A* is a *k*-vector space.

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- A homomorphism of k -algebras $\phi : A \rightarrow B$ is a ring homomorphism that is k -linear.

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- A homomorphism of *k*-algebras $\phi : A \rightarrow B$ is a ring homomorphism that is *k*-linear.
- A ring *R* is reduced if it has no nilpotent elements. That is, if $f \in R$ and $f^q = 0$ for some $q > 0$, then $f = 0$.

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- A ring R is reduced if it has no nilpotent elements. That is, if $f \in R$ and $f^q = 0$ for some $q > 0$, then $f = 0$.
- A k -algebra A is finitely generated (as a k -algebra) if there is a surjective k -algebra homomorphism

$$k[T_1, \dots, T_n] \twoheadrightarrow A.$$

Some algebra

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- A k -algebra A is finitely generated (as a k -algebra) if there is a surjective k -algebra homomorphism

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Equivalently, if there are elements $g_1, \dots, g_n \in A$ such that every element of A can be represented as a polynomial in the g_i 's with k -coefficients.

Algebra Examples

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Connection

J. David
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Parallels

- The ring

$$A := \frac{k[T_1, T_2, T_3]}{(T_1 T_2, T_3^2 + T_2^3, T_1 T_3, T_2 T_3)}$$

is a finitely generated k -algebra.

Algebra Examples

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- The ring map $k[T_1, T_2, T_3] \rightarrow A : T_i \mapsto T_i$ is a surjective k -algebra homomorphism.

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is a finitely generated k -algebra.

- The ring map $k[T_1, T_2, T_3] \rightarrow A : T_i \mapsto T_i$ is a surjective k -algebra homomorphism.
- $k[T]/(T^5)$ is not reduced but $k[T_1, T_2]/(T_1 + T_2 + 1)$ is.

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Theorem:

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Theorem:

- Let AAS be the category of affine algebraic sets over k with regular morphisms.

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Theorem:

- Let AAS be the category of affine algebraic sets over k with regular morphisms.
- Let FGR be the category of finitely generated reduced k -algebras with k -algebra homomorphisms.

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Theorem:

- Let AAS be the category of affine algebraic sets over k with regular morphisms.
- Let FGR be the category of finitely generated reduced k -algebras with k -algebra homomorphisms.
- There is an anti-equivalence of categories $\mathcal{E} : AAS \rightarrow FGR$.

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- Let FGR be the category of finitely generated reduced k -algebras with k -algebra homomorphisms.
- There is an anti-equivalence of categories $\mathcal{E} : AAS \rightarrow FGR$.
- I'm going to spell it out in detail.

Algebraic Correspondence: $\text{Sets} \rightarrow \text{Rings}$

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Parallels

- Let X be an affine algebraic set.

Algebraic Correspondence: Sets \rightarrow Rings

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Parallels

- Let X be an affine algebraic set.
- Define the coordinate ring of X as the ring of regular functions on X under pointwise addition and multiplication. It is written $A[X]$.

Algebraic Correspondence: Sets \rightarrow Rings

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Parallels

- Let X be an affine algebraic set.
- Define the coordinate ring of X as the ring of regular functions on X under pointwise addition and multiplication. It is written $A[X]$.
- $A[X]$ is a reduced k -algebra.

Algebraic Correspondence: $\text{Sets} \rightarrow \text{Rings}$

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Parallels

- Let X be an affine algebraic set.
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- $A[X]$ is a reduced k -algebra.
- The Weak Nullstellensatz implies that $A[X]$ is finitely generated.

Algebraic Correspondence: Sets \rightarrow Rings

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- $A[X]$ is a reduced k -algebra.
- The Weak Nullstellensatz implies that $A[X]$ is finitely generated.
- $\mathcal{E}(X) := A[X]$

Algebraic Correspondence: Morphisms \rightarrow Homs

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Parallels

- Let $\psi : X \rightarrow Y$ be a regular morphism.

Algebraic Correspondence: Morphisms \rightarrow Homs

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Parallels

- Let $\psi : X \rightarrow Y$ be a regular morphism.
- If $f \in A[Y]$, then $\psi^*f : X \xrightarrow{\psi} Y \xrightarrow{f} \mathbb{A}^1(k)$ is a regular morphism, so $\psi^*f \in A[X]$.

Algebraic Correspondence: Morphisms \rightarrow Homs

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- In fact, $\psi^* : A[Y] \rightarrow A[X]$ is a k -algebra homomorphism.

Algebraic Correspondence: Morphisms \rightarrow Homs

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- $\mathcal{E}(\psi) := \psi^*$

Algebraic Correspondence: Rings \rightarrow Sets

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Parallels

- Let A be a finitely generated reduced k -algebra.

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Parallels

- Let A be a finitely generated reduced k -algebra.
- Let $\rho : k[T_1, \dots, T_n] \twoheadrightarrow A$ be a surjective k -algebra homomorphism.

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Parallels

- Let A be a finitely generated reduced k -algebra.
- Let $\rho : k[T_1, \dots, T_n] \twoheadrightarrow A$ be a surjective k -algebra homomorphism.
- $\ker \rho$ is a finitely generated ideal of $k[T_1, \dots, T_n]$ by Hilbert's Basis Theorem.

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- Let $X \subseteq \mathbb{A}^n(k)$ be the affine algebraic set determined by a set of generators of $\ker \rho$.

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- Let $X \subseteq \mathbb{A}^n(k)$ be the affine algebraic set determined by a set of generators of $\ker \rho$.
- $\mathcal{E}^{-1}(A) := X$.

Algebraic Correspondence: $\text{Hom}_k \rightarrow \text{Morphisms}$

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Parallels

- Let $h : B \rightarrow A$ be a homomorphism of finitely generated reduced k -algebras.

Algebraic Correspondence: $\text{Hom}_k \rightarrow \text{Morphisms}$

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Parallels

- Let $h : B \rightarrow A$ be a homomorphism of finitely generated reduced k -algebras.
- $A \cong k[T_1, \dots, T_n]/I$ and $B \cong k[S_1, \dots, S_m]/J$

Algebraic Correspondence: $\text{Hom}_k \rightarrow \text{Morphisms}$

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- $A \cong k[T_1, \dots, T_n]/I$ and $B \cong k[S_1, \dots, S_m]/J$
- $h(S_i) = P_i(T_1, \dots, T_n) \pmod I$ for all i and some polynomials $P_1, \dots, P_m \in k[T_1, \dots, T_n]$.

Algebraic Correspondence: $\text{Hom}_k \rightarrow \text{Morphisms}$

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- Let $\psi : X \rightarrow Y$ be given by $\psi(\mathbf{a}) = (P_1(\mathbf{a}), \dots, P_m(\mathbf{a}))$.

Algebraic Correspondence: $\text{Hom}_k \rightarrow \text{Morphisms}$

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- $\mathcal{E}^{-1}(h) := \psi$

Galois Connection

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Parallels

- Let X be an affine algebraic set with coordinate ring A .

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Parallels

- Let X be an affine algebraic set with coordinate ring A .
- Let $Y \subseteq X$. Define $I(Y)$ as the ideal of functions that vanish on Y .

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Parallels

- Let X be an affine algebraic set with coordinate ring A .
- Let $Y \subseteq X$. Define $I(Y)$ as the ideal of functions that vanish on Y .
- Let $S \subseteq A$. Define $V(S)$ as the solution set of S .

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Parallels

- Let X be an affine algebraic set with coordinate ring A .
- There is a pair of inclusion reversing functions

$$\mathcal{I} : \mathcal{P}(X) \rightarrow \mathcal{P}(A) \text{ and } \mathcal{V} : \mathcal{P}(A) \rightarrow \mathcal{P}(X)$$

such that for all $Z \in \mathcal{P}(X)$ and $J \in \mathcal{P}(A)$:

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- $J \subseteq \mathcal{I}(Z)$ iff $Z \subseteq \mathcal{V}(J)$ (Galois Connection)

Galois Connection

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- $J \subseteq \mathcal{I}(Z)$ iff $Z \subseteq \mathcal{V}(J)$ (Galois Connection)
- $\mathcal{I}(\mathcal{V}(J)) = \sqrt{(J)}$ (Weak Nullstellensatz)

Galois Connection

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- $\mathcal{I}(\mathcal{V}(J)) = \sqrt{(J)}$ (Weak Nullstellensatz)
- $\mathcal{V}(\mathcal{I}(Z)) = \overline{Z}$

Consequences

- closed sets vs radical ideals

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- closed sets vs radical ideals
- irreducible closed sets vs prime ideals

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Consequences

- closed sets vs radical ideals
- irreducible closed sets vs prime ideals
- points vs maximal ideals

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- closed sets vs radical ideals
- irreducible closed sets vs prime ideals
- points vs maximal ideals
- dimensions match

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- closed sets vs radical ideals
- irreducible closed sets vs prime ideals
- points vs maximal ideals
- dimensions match
- varieties vs field extensions

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- closed sets vs radical ideals
- irreducible closed sets vs prime ideals
- points vs maximal ideals
- dimensions match
- varieties vs field extensions
- compact Riemann surfaces vs finite extensions of $\mathbb{C}(t)$

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- closed sets vs radical ideals
- irreducible closed sets vs prime ideals
- points vs maximal ideals
- dimensions match
- varieties vs field extensions
- compact Riemann surfaces vs finite extensions of $\mathbb{C}(t)$
- calculate tangent spaces with commutative algebra

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Parallels

- closed sets vs radical ideals
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- compact Riemann surfaces vs finite extensions of $\mathbb{C}(t)$
- calculate tangent spaces with commutative algebra
- classify singularities with commutative algebra (e.g. $k[[T_1, T_2]]/(T_1 T_2)$)

Consequences

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Parallels

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- calculate tangent spaces with commutative algebra
- classify singularities with commutative algebra (e.g. $k[[T_1, T_2]]/(T_1 T_2)$)
- vector bundles vs finitely generated projective modules

Consequences

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Parallels

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- compact Riemann surfaces vs finite extensions of $\mathbb{C}(t)$
- calculate tangent spaces with commutative algebra
- classify singularities with commutative algebra (e.g. $k[[T_1, T_2]]/(T_1 T_2)$)
- vector bundles vs finitely generated projective modules
- etc.