- The purpose of this note is to show that proper normal subgroups of S_n are contained in the alternating subgroup for $n \geq 1$. The cases of $n = 1, 2$ are both trivial. I will prove the contrapositive for the case $n \geq 3$.
- Suppose that H is a normal subgroup of S_n not contained in the alternating group and let $\sigma \in H \setminus A_n$. By the second isomorphism theorem, $H \cap A_n$ is a normal subgroup of A_n.
- Suppose $n \neq 4$. Then A_n is simple, and $H \cap A_n \in \{1, A_n\}$ as a consequence.
- If $H \cap A_n = 1$, then $\sigma^2 = 1$ because σ^2 is an even permutation contained in H. Therefore, σ is a product of disjoint transpositions. If (ij) occurs in the cycle decomposition of σ, then there is $\tau \in S_n$ such that (ik) occurs in the cycle decomposition of $\tau\sigma\tau^{-1} \in H$ for some $k \neq i, j$. This is because H is normal, $n \geq 3$, and all permutations of the same cycle decomposition are conjugate in S_n. Furthermore, $\tau\sigma\tau^{-1} \notin A_n$ because A_n is normal and $\sigma \notin A_n$. By construction, $(\tau\sigma\tau^{-1})\sigma$ takes j to k, and is therefore a non-trivial element. On the other hand it is even because it is the product of two odd permutations. So $H \cap A_n$ has a non-trivial element, in contradiction to our hypothesis. Thus $A_n \subseteq H$ (i.e. $H \cap A_n = A_n$), and by hypothesis that are not equal. By Lagrange's theorem, the order of H divides $n!$ (the order of S_n) and is a strict multiple of $n!/2$ (the order of A_n). Therefore, $H = S_n$.
- Suppose that $n = 4$. By the same argument as for general $n \geq 3$, we can see that $H \cap A_4$ is a non-trivial group. Let $\rho \in H \cap A_4, \rho \neq 1$. If ρ is a 3-cycle, then H contains all 3-cycles (because they are conjugate). Since A_n is generated by 3-cycles, $A_4 < H$. The inclusion is strict because $\sigma \in H$. Suppose ρ is a product of two disjoint transpositions. Looking at partitions of 4 shows that σ is either a a transposition or a 4-cycle. If σ is a transposition, then since H is a normal subgroup of S_n all transpositions are conjugate, and they generate S_n, we have $H = S_n$. Thus we may suppose that σ is a 4-cycle. Therefore, H contains all 4-cycles (there are 6 of them), each product of 2 disjoint permutations (there are 3 of them), and the identity. So $|H| \geq 10$ and $|H|$ divides 24. By the pigeon hole principle, there is $\tau \in H$ such that τ is either a transposition or a 3-cycle. In either case, H strictly contains A_4.